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Contents

Chapter 1

Habit Formation: Implications for Investors[1]

This chapter reviews the literature on habit formation and primarily focuses on the implications for investors. Habit formation utility preferences differ from the traditional ones in that they relax the assumption of time-separability. This realistic feature has substantial impact on the optimal portfolio and consumption strategy of investors. First, it induces a new subsistence portfolio that ensures future habit consumption. Second, the equity investment is dampened by habit persistence. Third, the optimal consumption strategy is decomposed into two components: one is the subsistence consumption and the other is linked to the returns of risky investment. Fourth, habit formation utility preferences result in less consumption smoothing than time-separable utility preferences.

1.1 Introduction

Time separable utility functions, such as power utility, are common in the asset pricing and portfolio choice literature. However, it is widely acknowledged that the assumption of time separability makes it difficult for traditional models to reproduce the empirical regularities of asset returns and households' consumption and investment behavior. To this end, alternative utility functions without time separability have been proposed and gained popularity in recent years. Preferences with habit persistence are prominent in this literature. Specifically, such preferences prescribe that the investors derive utility

[1]This paper is based on De Jong and Zhou (2014).

only from the consumption on top of habit levels. The academic literature has shown habit-based models are useful in resolving a number of asset pricing anomalies and the asset allocation puzzle of households. In this chapter, we review the major contributions on habit formation in the literature and focus particularly on the implications of habit formation for optimal portfolio and consumption choice.

We first discuss the features of habit formation models, which differ from each other with respect to how the utility function is formulated and how the habit is formed. On the one hand, depending on how the surplus consumption is defined, there are two types of habit-based utility functions: one is "ratio habit model", where surplus consumption is given by the *ratio* between consumption and the habit level and the other one is "difference habit model", where surplus consumption is given by the *difference* between consumption and the habit level. On the other hand, there is a distinction between "internal habit formation" and "external habit formation". The habit level depends on an individual's own consumption in the former case, but on the past history of aggregate consumption in the latter case.

Because habit formation utility preferences can generate time-varying risk aversion, they have proved successful in explaining a wide range of asset pricing anomalies, such as the equity premium puzzle, the failure of expectation hypothesis and the uncovered interest rate parity puzzle. By contrast, the empirical evidence for habit formation is rather mixed.

The presence of habit formation has a significant effect on the optimal portfolio and consumption strategy. The optimal consumption strategy is split into two components: one is the subsistence consumption and the other component of consumption is linked to the returns of risky investment. Habit formation utility preferences induce stronger saving motive, a lower consumption rate in the early periods and produce higher consumption growth. As a consequence, habit formation consumption policies result in less consumption smoothing than time-separable utility preferences. This effect is less pronounced for the habit-investors who allow their habit consumption to be eroded by inflation. In the investment policy for habit-investors, there should be a clear separation of the subsistence portfolio and the traditional portfolios. In the subsistence portfolio, investors invest only in long-term bonds to ensure future subsistence consumption. The investments in equity are dampened by habit formation, because equity is appropriate neither for interest rate hedging nor for ensuring future subsistence consumption level.

For investors with low wealth, this dampening effect leads to much more conservative equity investment decisions, which may provide an explanation for some asset allocation puzzles.

The structure of this paper is organized as follows. Section 1.2 discusses different types of habit formation models, their success in the asset pricing literature and the empirical evidence for/against them. Section 1.3 investigates the optimal portfolio and consumption strategy for habit-investors in a variety of setting. Section 1.4 summarizes the implications of habit formation for investors.

1.2 Habit Formation Models

1.2.1 Types of Habit Formation Models

Time-separability is a conventional assumption for utility functions in financial economics. It implies that the marginal rate of substitution between any two periods is independent of the consumption in any other period and the consumption in a certain period does not have a direct influence on the utility in any other periods. In a standard life-cycle model with a time-separable utility preference, the objective function of an agent with a fixed investment horizon T in a continuous-time setting can be written as

$$\max E\left[\int_0^T e^{-\delta t}U(t,c_t)\mathrm{d}t + e^{-\delta T}B(w_T)\right], \tag{1.1}$$

where δ is the subjective time discount factor, c_t is the consumption at time t, $B(\cdot)$ is the bequest function, and w_T is the bequeathed wealth. The utility function $U(\cdot)$ typically takes the form of Constant Relative Risk Aversion (CRRA)

$$U(t,c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \tag{1.2}$$

where γ is the coefficient of relative risk aversion.

In contrast to the prevailing time separable utility functions, those with habit formation assume that utility in the current period depends not only on the consumption in the same period but also on the consumption in the previous periods or the past history of aggregate consumption. Typically, the utility optimization problem for a finite horizon investor with habit persistence is formulated as

$$\max E\left[\int_0^T e^{-\delta t}U(t, c_t, h_t)\mathrm{d}t + e^{-\delta T}B(w_T)\right], \tag{1.3}$$

where h is the habit level. Equation (1.3) shows that in the habit formation models, the instantaneous utility depends not only on the current consumption but also on the habit level. In the "ratio habit model", utility depends on the ratio of current consumption c_t to the habit level h_t. Specifically, the utility function is

$$U_t \;=\; \frac{(c_t/h_t)^{1-\gamma}}{1-\gamma}, \tag{1.4}$$

In this specification, the relative risk aversion $R(c) = -cU''/U'$ equals the coefficient γ and does not depend on the habit level. As $R(c)$ essentially determines the asset allocation, the ratio habit utility function has little impact on the asset allocation (although it does affect savings behavior over the life cycle). Instead, in this paper we use the so-called "difference habit model", where surplus consumption is given by the *difference* between consumption and the habit level. Specifically, the utility function is

$$U_t \;=\; \frac{(c_t-h_t)^{1-\gamma}}{1-\gamma}, \tag{1.5}$$

where $c_t - h_t$ is the surplus consumption level. In this specification, the relative risk aversion is $R(c) = \gamma\frac{c}{c-h}$ and varies with the habit level: the closer current consumption is to the habit, the more risk averse the investor.

In terms of how habit is formed, there is a distinction between "internal habit formation" and "external habit formation". Constantinides (1990) considers a linear internal habit formation process[2]:

$$h_t = h_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)} c_s \mathrm{d}s, \tag{1.6}$$

where c is the investor's own consumption. α determines how strongly past consumption affects current consumption and is called the scaling parameter. β determines how fast the effect of previous consumption on the habit level diminishes and is called the

[2]Other papers using internal habit formation models include Ryder and Heal (1973), Sundaresan (1989), Munk (2008) and De Jong and Zhou (2013b).

persistence parameter. h_0 is the initial habit level. The habit level is a weighted average of past consumption rates. Note that the weights are exponentially decreasing so that the recent consumption receives a higher weight. As the habit level is linear in the previous consumptions, this type of habit is referred to as "linear habit formation". It is easy to see that when $\alpha = \beta = 0$, the model reduces to a time-separable model. Taking the derivative of (1.6) with respect to time t yields

$$\mathrm{d}h_t = -(\beta h_t - \alpha c_t)\mathrm{d}t. \tag{1.7}$$

When $c_t = h_t$, $\mathrm{d}h_t = -(\beta - \alpha)h_t\mathrm{d}t$. Thus, $(\beta - \alpha)$ can be interpreted as the decay rate of habit level at the minimum consumption and captures habit strength.

In contrast, in the external habit formation models, the habit level depends on the past history of aggregate consumption; that is, habit formation is an externality. It is also referred to as "catching up with Joneses". Abel (1990) considers an external habit formation model[3]:

$$h_t = \left(c_{t-1}^{\beta}\bar{c}_{t-1}^{1-\beta}\right)^{\alpha}, \tag{1.8}$$

where $\bar{c}$ is the aggregate consumption. In this model, the habit level depends not only on the investor's own consumption, but also on the aggregate consumption, which is specified exogenously.

Although both internal and external habit formation models are utilized in the asset pricing literature, to the best of our knowledge none of the extant literature on the portfolio choice employs external habit model. Gomes and Michaelides (2003) indicate that considering external habits in a partial equilibrium framework would be difficult, since the aggregate consumption process can not be taken as exogenous. Endogenous aggregate consumption leads to an endogenous evolution of the habit and it is not obvious how agents form expectations about the future evolution of this habit in equilibrium. Therefore, we narrow our focus to internal habit formation in the following section on portfolio choice.

Finally, another simple but extreme form of habit formation is the so-called ratchet consumption, which requires that consumption will never decrease over time. Scott

[3]Other papers studying models with external habit formation include Gali (1994), Campbell and Cochrane (1999) and Wachter (2006).

and Watson (2011) propose a rule of thumb—the Floor-Leverage rule for retirees with ratchet consumption preferences. According to this rule, retirees should set up a *floor portfolio* comprised of the risk-free asset using at least 85% of the wealth and a *surplus portfolio* comprised of a leveraged position in risky assets using the remaining wealth. This ratchet consumption model was analyzed in Chapter 2, and we refer to that for details.

1.2.2 Asset Pricing with Habit Formation

Habit formation models have gained popularity in recent years and in particular have become increasingly successful and important in explaining a wide variety of asset pricing phenomena. Sundaresan (1989) and Constantinides (1990) show that habit formation models can be use to rationalize a high equity premium with low levels of risk aversion. Campbell and Cochrane (1999) formulate a model with habit formation captured by the so-called surplus consumption ratio, which is assumed to be slow-moving and thereby generates time variation in price of risk. Armed with the slow countercyclical variation in Sharpe ratio, their model explains the equity premium puzzle as well as a number of asset pricing facts. Following the specification of habit persistence in Campbell and Cochrane (1999), Wachter (2006) establishes a model that produces realistic means and volatilities of bond yields and accounts for the expectations puzzle. Verdelhan (2010) uses a similar framework to resolve the uncovered interest rate parity puzzle.

1.2.3 Empirical Evidence on the Existence of Habit Formation in Consumption

The presence of habit formation in consumption has been extensively tested in the literature and in general mixed conclusions have been drawn. Most of the tests are performed on the basis of the Euler equation derived from the habit-based utility functions. Ferson and Constantinides (1991) find evidence in monthly, quarterly, and annual aggregate consumption data that habit persistence exists. Heien and Durham (1991) test the linear habit formation hypothesis that the habit is proportionate to past consumption and show that habit effects are highly significant. Korniotis (2010) jointly tests the presence of internal and external habit formation using the U.S. state level consumption data and find evidence in support of the latter one. On the contrary, Meghir and Weber

(1996) show that there is no empirical support for intertemporal non-separability of preferences. Dynan (2000) first shows that a simple model of habit formation implies a condition relating the strength of habits to the evolution of consumption and estimates this condition with the U.S. food consumption data. The results yield no evidence of habit formation at the annual frequency.

1.3 Consumption and Portfolio Choice with Habit Formation

This section is devoted to the discussion of the implications of habit formation for optimal consumption and portfolio choice. This section is organized as follows. Subsection 1.3.1 and 1.3.2 review Merton's life-cycle model and the linear habit formation model under the assumption of constant investment opportunities. Subsection 1.3.3 and 1.3.4 discuss the effects of time-varying investment opportunities and inflation risk on the asset allocation strategy in the linear habit formation framework. Subsection 1.3.5 discusses how labor income affects the optimal portfolio strategy of habit-investors.

1.3.1 Benchmark: Merton's Model

Merton (1969) solves the life-cycle model of portfolio and consumption choice in a continuous-time setting with time-separable utility. The objective of the investor is characterized by the CRRA utility function specified in equations (1.1) and (1.2). There are two assets available to the investor, a risky asset with constant expected return of μ and volatility of σ and a riskless asset that carries a fixed interest rate of r. Then, the returns of these two assets follow diffusion processes:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}z_t, \tag{1.9}$$

$$\frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t, \tag{1.10}$$

where S and B denote the price of the risky asset and the price of the risk-free asset respectively.

Solving the model using dynamic programming approach yields

$$x_t^* = \frac{\lambda}{\gamma\sigma}, \tag{1.11}$$

$$c_t^* = \left[\frac{v}{1-e^{v(t-T)}}\right] w_t, \tag{1.12}$$

with

$$v = \frac{1}{\gamma}\left[\delta + (\gamma-1)\left(\frac{\lambda^2}{2\gamma}+r\right)\right], \tag{1.13}$$

$$\lambda = \frac{\mu - r}{\sigma}, \tag{1.14}$$

where w is the wealth level, x is the fraction of wealth invested in the risky asset and λ is the Sharpe ratio. Equations (1.11) and (1.12) correspond to the optimal portfolio and consumption strategy, respectively. It is worth noting that the optimal portfolio strategy is independent of wealth and investment horizon, which conflicts with the conventional wisdom that the allocation to stock should decrease with age. This follows from the fact that facing constant investment opportunities, the investor is myopic and holds risky assets only for the speculative purpose.

The optimal consumption strategy, as shown in Equation (1.12), is to consume a fraction of current wealth. This fraction only depends on the remaining lifetime $T-t$ and can be interpreted as the annuity value with an interest rate ν. This consumption function implies that a negative shock to wealth is translated to a lower consumption now and in the future. There is no guaranteed minimum consumption level. Because the stock market risk makes the wealth volatile, the consumption level fluctuates over time. For illustrative purpose, we present a numerical example. Consider a retiree with age of 65, risk aversion of $\gamma = 3.5$, a 20-year horizon, a subjective discount factor of $\delta = 0.05$ and initial wealth of $\$10,000$. The riskfree rate and equity risk premium are set to 2% and 4% respectively. The volatility of the stock is 18%. With this calibration, it is easy to determine the constant investment strategy in Merton's model: $x = 35.3\%$. Figure 1.1 shows the expected and sample consumption paths. While the expected consumption is smooth over time, the sample consumption fluctuates remarkably and declines sharply in some periods. Apparently, a habit-investor will suffer substantially from these intermittent big drops in consumption. Therefore, the optimal consumption strategy in Merton's model does not fit the need of investors with habit persistence.

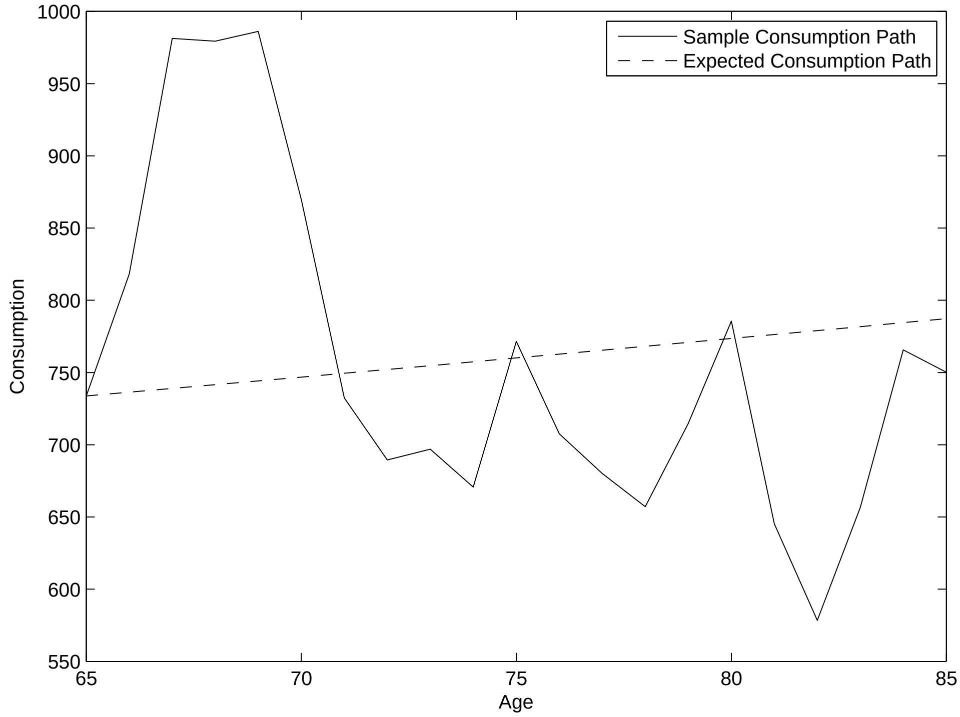


Figure 1.1: Expected and sample consumption paths in Merton's model and linear habit formation model.

1.3.2 Linear Habit Formation Model

The linear habit formation life-cycle model is well formulated by Equations (1.3), (1.5) and (1.6). The investment opportunities are assumed constant as Merton's model. Munk (2008) shows that the optimal consumption and portfolio strategy are given by

$$c_t^* = h_t^* + (1 + \alpha F_t)^{-\frac{1}{\gamma}} \frac{w_t^* - h_t^* F_t}{G_t} \tag{1.15}$$

$$x_t^* = \frac{w_t^* - h_t^* F_t}{w_t^*} \frac{1}{\gamma} \sigma^{-1} \lambda, \tag{1.16}$$

where F is given by

$$F_t = \int_t^T e^{-(r+\beta-\alpha)(s-t)} \mathrm{d}s = \frac{1}{r+\beta-\alpha} \left(1 - e^{-(r+\beta-\alpha)(T-t)}\right), \tag{1.17}$$

and G is a deterministic function of time. F_t can be interpreted as the price of a bond paying a continuous coupon which is declining at exactly the decay rate of future habit levels, $(\beta - \alpha)$ and $h_t F_t$ is the cost of ensuring that future consumption never falls below

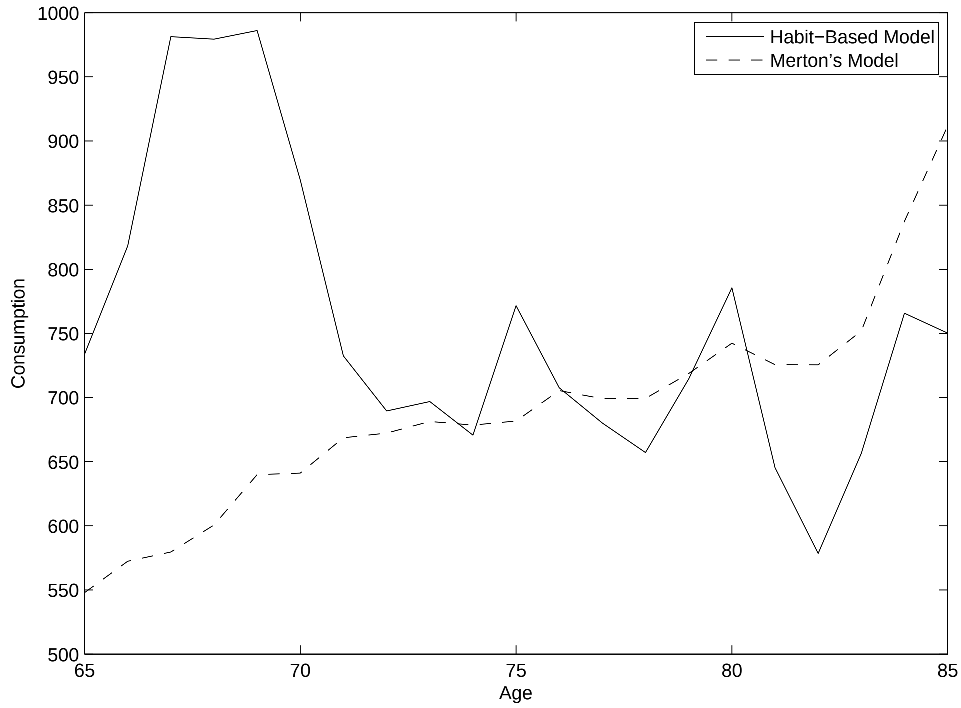


Figure 1.2: Sample Consumption: Merton's Model vs Linear Habit Formation Model.

the current habit.

Comparing (1.16) with (1.11) reveals that the optimal portfolio strategy is no longer constant; it becomes dependent on investment horizon, wealth and habit level. The optimal fraction of the free wealth $w-hF$ invested in the stock coincides with the optimal fraction of total wealth w for an investor in Merton's model. Habit persistence reduces risk-taking because in order to sustain consumption at habit level, the investor has to put aside an amount of wealth hF and invest it in risk-free asset. As a consequence, the wealth that can be freely invested reduces to $w - hF$. The optimal portfolio weight of the risk asset, x, decreases with the investment horizon, since longer horizon induces the investor to reserve more money to ensure that future consumption always exceeds the habit level. Therefore, the habit formation model implies an optimal portfolio strategy that contradicts the popular advice that older investors should be more conservative than young investors. However, the dampening effect of habit formation diminishes for richer investor: as wealth goes to infinity, x increases to the level that is optimal in Merton's model.

The optimal consumption strategy in (1.15) is to consume the current habit level h plus a time and state-dependent fraction of the free wealth $W - hF$. In contrast, in

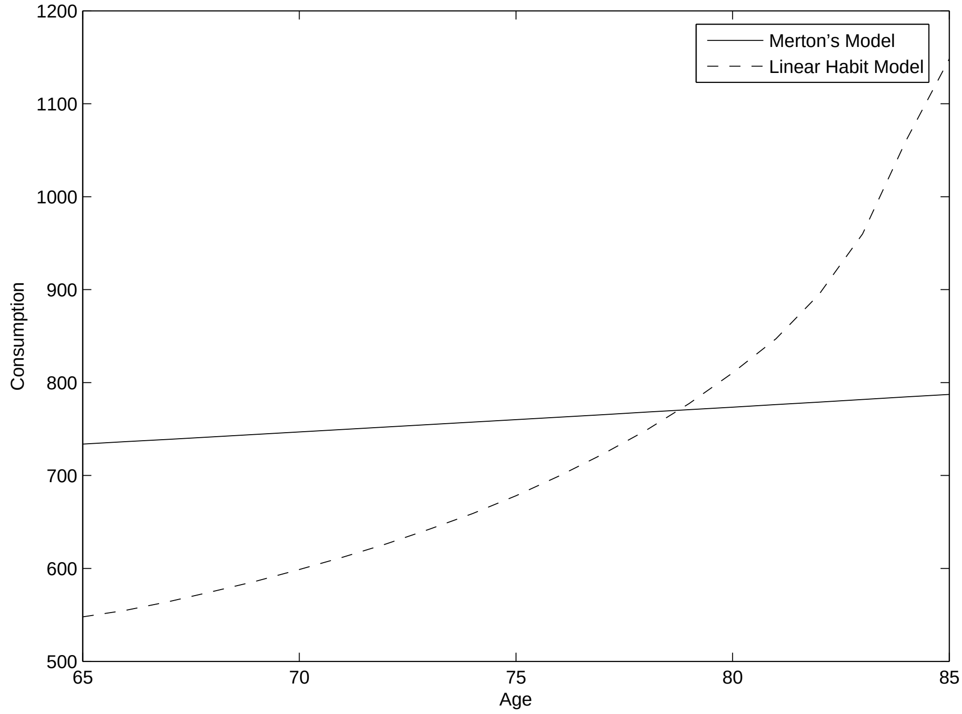


Figure 1.3: Expected Consumption: Merton's Model vs Linear Habit Formation Model.

Merton's model, the investor consumes a time-dependent fraction of the total wealth, which is clearly shown in (1.12). The distinctions between the optimal consumption strategy in the two models can be attributed to habit constraints: to ensure that future consumption can meet the habit level, the investor must first consume the current habit level in each period and then consume a time-dependent fraction of the free wealth.

Figure 1.2 contrasts a sample consumption path in Merton's model with that in linear habit formation model. Obviously, the habit investor's sample consumption exhibits an increasing trend, while the non-habit investor's sample consumption goes up and down over the life-cycle. Figure 1.3 illustrates the expected consumption in Merton's model and linear habit formation model. Compared with the habit investors, the non-habit investor does not have to reserve a fraction of wealth for ensuring future subsistence consumption and enjoy higher consumption in the early periods, which is clearly displayed in the right graph. In later periods, however, the consumption of the habit investors exceeds that of the non-habit investor because of the higher saving rate generated by habit formation. This difference implies that habit formation leads to less consumption smoothing.

We can summarize the implications of the linear habit formation model under constant investment opportunities as follows. First, in order to meet the habit constraints, habit-investors should reserve a certain amount of wealth and invest it in habit bonds, the price of which is dependent on the investment horizon and habit strength. Then, the rest of the wealth can be viewed as free wealth and invested in a traditional fashion, such as the portfolio rule suggested by Merton's model. The optimal consumption strategy is to consume the habit level (determined by previous consumption choices) plus a fraction of current wealth. In contrast to the Merton's model, there is a guaranteed minimum consumption level equal to the habit. The consumers saving and investment policy has to make sure that this habit level can always be consumed.

1.3.3 Stochastic Investment Opportunities

The assumption of constant investment opportunities is undoubtedly restrictive. There is ample empirical evidence that that stock returns and short-term interest rates are time-varying and mean reverting[4], which implies that μ in (1.9) and r in (1.10) are not constant but dependent on time and states.

Munk (2008) examines the cases with mean reversion in stock returns and stochastic interest rates in the linear habit formation model specified above. Munk (2008) shows that the optimal fraction of wealth invested in stocks is the sum of a myopic demand and a (positive) hedge demand. Habit persistence has different effects on these two components, but the differences are very small. Contrary to the case of time-additive utility, the optimal fraction of wealth invested in stocks is not necessarily monotonically decreasing over the life of an investor with habit persistence in preferences for consumption.

Another type of variation in the investment opportunities is stochastic interest rate. Munk (2008) assumes that interest rates evolve according to the CIR model

$$\mathrm{d}r_t = \kappa(\bar{r} - r_t)\mathrm{d}t - \sigma_r\sqrt{r_t}\mathrm{d}z_{1t}, \tag{1.18}$$

[4]For mean reversion in stock returns, see Poterba and Summers (1988) and Fama and French (1989). For mean reversion in short-term interest rates, see Wu and Zhang (1996), Wu and Chen (2001) and Seo (2003)

and the dynamics of the bond price maturing at time t (P_t^s) and stock prices (S_t) are

$$\frac{\mathrm{d}P_t^s}{P_t^s} = (r_t + B(s-t)\lambda_1 r_t)\,\mathrm{d}t + B(s-t)\sigma_r\sqrt{r_t}\mathrm{d}z_{1t}, \tag{1.19}$$

$$\frac{\mathrm{d}S_t}{S_t} = (r_t + \sigma\psi(r_t))\,\mathrm{d}t + \sigma\rho\mathrm{d}z_{1t} + \sigma\sqrt{1-\rho^2}\mathrm{d}z_{2t}, \tag{1.20}$$

where z_1 and z_2 are two one-dimensional standard Brownian motions independent of each other and ρ is the instantaneous correlation between stock returns and bond returns. λ_1 is the market price of risk associated with z_1. B is a function of time and ψ is a function of the short rate. Equation (1.18) shows that to model the short rate, another source of uncertainty z_1 is introduced. As a result, another asset, namely bond, is added to the asset menu in order to complete the market. It is important to note that in the economy with interest rate risk, bonds, rather than cash, are risk-free assets for investors, whose horizon aligns with maturity of the bond.

In this setting, the price of the habit bond[5] F becomes

$$F(t,r) = \int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} P_t^s \mathrm{d}s. \tag{1.21}$$

Comparing with (1.17) shows that the habit bond consists of a series of zero-coupon bonds rather than instantaneously risk-free asset (cash) because of the interest rate risk. When $\rho = 0$, the optimal portfolio strategy is given by

$$\begin{aligned} x_t^{P*} &= \frac{w_t^* - h_t^* F_t}{w_t^*}\frac{1}{\gamma\sigma_r B(T-t)}\frac{\lambda_1}{\sigma_r} - \frac{w_t^* - h_t^* F_t}{w_t^*}\frac{1}{B(T-t)}\frac{(\partial G/\partial r)\,(t,r)}{G(t,r)} \\ &\quad + \frac{h_t^*}{w_t^*}\frac{(\partial F/\partial r_t)\,(t,r_t)}{F(t,r_t)}, \end{aligned} \tag{1.22}$$

$$x_t^{S*} = \frac{w_t^* - h_t F_t}{w_t^*}\frac{\lambda_2}{\gamma\sigma_S}, \tag{1.23}$$

where λ_2 is the market price of risk associated with z_2 and G is a function of time and interest rate. x^{P*} and x^{S*} are the optimal allocation to the bond and stock, respectively. x^{P*} consists of three components: a myopic portfolio that invests in the mean-variance tangency portfolio, a hedge portfolio that provides hedge against variation of future investment opportunities in the economy modified by the presence of habit formation,

[5] Habit bond is defined as a bond paying continuous coupons which are declining at the decay rate of habit levels ($\beta - \alpha$).

and a subsistence portfolio that ensures that the future subsistence consumption level can be satisfied. It can be proved G is decreasing in r so that the fraction of wealth invested in the hedge portfolio is positive, which is consistent with the intuition that the holding bond allows investors to hedge interest risk. As in the previous case, both the myopic portfolio and the hedge portfolio are dampened by habit persistence. The variation in interest rate generates interest rate hedge term in the subsistence portfolio, because in an economy with interest rate risk the future habit consumption is ensured with a (dynamically rebalanced) combination of the bonds. In contrast, the optimal stock investment only contains a myopic term since stocks are inappropriate either for interest rate hedging or for ensuring future subsistence consumption level. It is also lowered by habit persistence.

Several implications can be drawn from the cases with stochastic investment opportunities. First, habit-investors should set up a hedge portfolio to hedge against adverse variation in future investment opportunities. Second, in the presence of interest risk, bonds rather than cash should be used to ensure the future subsistence consumption level.

1.3.4 Inflation Risk

Hedging inflation risk is of great importance for long-term investors, such as individual investors and pension funds, as inflation substantially erodes the purchasing power of their wealth. For habit-investors, the interaction between the need to sustain future minimum consumption and the desire to hedge inflation risk may have a large impact on their optimal portfolio strategy. Therefore, it is of interest to incorporate inflation risk in the habit-based life-cycle model.

Chapter 3 investigates the optimal portfolio and consumption strategy in a life-cycle model with linear habit formation under inflation risk. We follow Brennan and Xia (2002) to model inflation dynamics:

$$\mathrm{d}\pi_t = \kappa_\pi(\bar{\pi} - \pi_t)\mathrm{d}t + \sigma_\pi \mathrm{d}z_{\pi t}, \tag{1.24}$$

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \mathrm{d}t + \xi' \mathrm{d}z_t, \tag{1.25}$$

where π is the instantaneous expected inflation, Π is the commodity price level and

ξ is the unexpected inflation shocks. The expected inflation follows a mean-reverting process and the realized inflation equals the expected inflation plus an i.i.d. unexpected inflation shock. The real habit process is similar to (1.6), except that c and h are taken as real consumption and real habit level, respectively. This implies that the real habit level is generated by past real consumption.

It is shown in Chapter 3 that the real prices of the habit bond under inflation is

$$f_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} \frac{m_s}{m_t} \mathrm{d}s \right] = \int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} p_t^s \mathrm{d}s, \tag{1.26}$$

where p is the price of inflation-indexed bond. Interestingly,under inflation risk the habit bond is comprised of inflation-indexed zero-coupon bonds rather than nominal bonds. This is consistent with Campbell and Viceira (2001) that long-term inflation-indexed bonds are the risk-free assets from long-term investors. Chapter 3 shows that the optimal portfolio strategy in complete market is

$$x_t^* = \frac{w_t^* - h_t^* f_t}{w_t^*} \frac{1}{\gamma} (\sigma')^{-1}(-\phi) + \frac{w_t^* - h_t^* f_t}{w_t^*} (\sigma')^{-1} \sigma_{gt} + \frac{h_t^* f_t}{w_t^*} (\sigma')^{-1} \sigma_{ft} + (\sigma')^{-1} \xi, \tag{1.27}$$

where ϕ captures the market prices of risks. Equation (1.27) expresses the optimal portfolio as the sum of four portfolios. The first portfolio (speculative portfolio) is proportional to the mean-variance tangency portfolio represented by $-(\sigma')^{-1}\phi$, which maximizes the instantaneous Sharpe ratio, and the amount invested in it is inversely related to the investor's relative risk aversion. The second portfolio hedges against variation in the investment opportunity set in the economy modified by the presence of habit formation. Both the speculative portfolio and hedge portfolio are dampened by the multiplier $(w-hf)/w$, because the habit investor has to set aside an amount of wealth hf for future minimum consumption stream. The third portfolio is a subsistence portfolio that invests in a coupon bond with continuous payments ensuring that the investor can meet his future minimum consumption process. While such a coupon bond may not be available on the market, it can be replicated by trading other bonds. The last portfolio is an inflation hedge portfolio that has a perfect correlation with the inflation realization. It is induced by the presence of inflation risk, because unexpected inflation shocks erode the value of portfolio and should be fully hedged away, if possible. Therefore, the key implication is that in the presence of inflation risk, habit-investors should achieve a full inflation hedge by holding inflation-indexed bonds. In the absence of index linked bonds, this latter part is replaced by a portfolio of assets that best replicates the return on an index linked bond, as in Brennan and Xia (2002).

Chapter 3 also considers a case in which the investor derives utility from consumption on top of real habit level, but forms habit on the basis of previous nominal consumption. This mismatch between utility function and habit formation process can be considered money illusion, because the investor confuses the nominal consumption stream with the real consumption stream in forming habit levels. As a consequence, the habit level is allowed to be eroded by inflation. It is shown in Chapter 3 that in the case of nominal habit formation, inflation risk plays a much bigger role in the case of nominal habit formation, because it alters the risk characteristics of both the hedge demand and subsistence demand and the inflation risk exposure of the overall portfolio is raised. Another distinction is that the subsistence portfolio is left uninsured because the subsistence consumption can be reduced by inflation. Moreover, the size of the subsistence portfolio shrinks and the dampening effect of habit persistence on risky investment is mitigated. The implication is that if the habit is formed in nominal terms, less money is reserved to ensure subsistence consumption and the portfolio allocation to risky assets is larger.

1.3.5 Labor Income and Asset Allocation Puzzle

In the above analysis, we have assumed that investor's wealth consists only of tradable financial assets. However, this is not a realistic description of the wealth of individual investors, since a large component of their wealth is the nontradable human wealth[6]. The nontradability generates two important features of human wealth that influence the consumption and asset allocation decisions of individual investors. First, labor income risk is uninsurable and idiosyncratic. This induces the investors to increase precautionary savings to hedge against future labor income shocks. Second, labor income can hardly be collateralized to finance current consumption and investment due to the moral hazard problem: having sold a claim against future income, an individual has no incentive to continue working. This is known as liquidity constraint in the literature on the life-cycle asset allocation. As both the level and risk of labor income vary over the life-cycle, age-dependent investment strategy can arise.

There is a large literature that studies the optimal consumption and investment

[6]Following Campbell and Viceira (2002), we define human wealth as the expected present discounted value of their future labor income.

strategy in the life-cycle models with stochastic uninsurable labor income[7]. However, these models are not able to match two important stylized facts: a low stock market participation rate and moderate equity holdings for households with equity investment. The failure follows from the fact that the calibrated correlation between labor income and stock returns is pretty low and therefore labor income resembles bonds rather than stocks. The models then predict that with large implicit holdings of bonds, investors are inclined to invest aggressively in stocks and the stockholdings should be higher for young investors than for older investors. This is known as the asset allocation puzzle. To resolve this puzzle, alternative models have been employed and a number of explanations have been proposed in the literature[8].

Motivated by the relative success of habit formation models in resolving asset pricing puzzles and modeling consumption dynamics, Gomes and Michaelides (2003) introduce habit formation preferences in a life-cycle model with uninsurable labor income risk. They find that the internal habit formation models have worse performance than their time-separable utility counterparts in matching the empirical regularities on asset allocation behavior. Because the presence of habit persistence leads to a stronger incentive to smooth consumption over time, investors accumulate more wealth early in life and have stronger motive to participate in stock markets. On the contrary, Polkovnichenko (2007) derives the habit-wealth feasibility constraints and focuses on the effect of low or even zero income realizations on portfolio allocation. He shows that when there is only a small probability of a disastrously low income, investors make much more conservative investment decisions because they have to satisfy the constraints that future habits implied by current consumption are feasible. The model predicts that for some low to moderately wealthy households, the allocation to stocks increases with wealth. Due to this relationship, the model can generate relatively more conservative portfolios

[7]Cocco, Gomes, and Maenhout (2005) is among the first to solves a life cycle model of consumption and portfolio choice with non-tradable labor income and borrowing constraints. Munk and Sørensen (2010) investigate the optimal investment and consumption choice of individual investors facing uncertain future labor income and stochastic interest rates. Van Hemert (2010) analyzes the mortgage and bond portfolio choice of household with stochastic labor income. Koijen, Nijman, and Werker (2010) study the importance of time-varying bond risk premia in a life-cycle model with labor income. Chapter 4 examines the inflation hedging power of human wealth in a life-cycle model from a cointegration point of view.

[8]Cocco (2005) shows housing investment is an important factor affecting stockholdings. Gomes and Michaelides (2005) establish that a model that features Epstein-Zin preferences, a fixed stock market entry cost, and moderate heterogeneity in risk aversion can generates predictions consistent with empirical observations. Benzoni, Collin-Dufresne, and Goldstein (2007) find that incorporating a cointegration relationship between dividend and labor income helps resolve the puzzle.

for young investors with lower savings, which is consistent with the empirical facts.

1.4 Concluding Remarks

This chapter reviews recent contributions on habit formation in the literature and investigates the implications of habit formation for investors. First, for habit-investors, there should be a clear separation of the subsistence portfolio and the speculative portfolio. In the subsistence portfolio, investors should invest in bonds to ensure the habit consumption. In contrast, they should invest more aggressively in the speculative portfolio to increase consumption rates. Second, the habit persistence constrains investors' risk taking behavior, and this effect is much more pronounced for young investors with low wealth. Third, in the presence of stochastic investment opportunities, habit-investors should set up a hedge portfolio to hedge changes in the investment opportunity set. To hedge interest rate risk, bonds should be a major asset in the hedge portfolio. Fourth, habit investors should achieve a full hedge against inflation by holding inflation-indexed bonds. Fifth, the optimal consumption should be split into two components: one finances the retirees' habit consumption and the other one is linked to the performance of the free funds.

Chapter 2

Guarantees and Habit Formation in Pension Schemes: A Critical Analysis of the Floor-Leverage Rule[1]

Scott and Watson (2011) have recently introduced a simple "floor-leverage" rule for investment when consumers never want to reduce consumption from one year to the next. We show that the leverage in their risky asset investment policy implies a positive probability of lower consumption than in the previous year. However, for realistically calibrated asset returns, insurance against such bankruptcy risk using put options (at the Black-Scholes prices) is inexpensive and can make the Floor-Leverage rule work. A comparison with standard life-cycle models of consumption and investment shows that the requirement of non-decreasing consumption is very costly in welfare terms, because it results in low early consumption and high consumption growth and contradicts the desire of households to smooth consumption over time from an economic point of view.

2.1 Introduction

Many pension plans in the Netherlands guarantee that the (nominal) benefits will never decrease. The benefits can increase if the financial position of the fund allows, according to the so-called conditional indexation rule. In exceptional circumstances, benefits can be cut ('afstempelen'), but this is a measure of last resort and considered to be very

[1]This chapter is based on De Jong and Zhou (2013a).

painful. In contrast to this policy, typical optimal consumption and investment models prescribe that consumption should always be adjusted to changes in wealth, without guarantees that consumption never decreases.

Recent academic literature suggests that investors regard a large part of their previous consumption as necessary for subsistence, and derive utility only from the excess of consumption above the subsistence level; this is referred to as habit formation.[2] As pension funds invest on behalf of their members, the habit formation of pension participants might have great impact on pension design and investment strategy of pension funds. As discussed above, many pension plans contain guarantees and habit formation might be a reason for the demand for such guarantees. Therefore, it is of interest to examine the impact of habit formation preferences on the optimal portfolio and consumption choice and explore the implications for pension funds.

A simple but extreme form of habit formation is the so-called ratchet consumption, which requires nondecreasing consumption over time. Scott and Watson (2011) analyze the portfolio choice problem with ratchet consumption constraint and propose a rule of thumb—the Floor-Leverage rule for retirement: to ensure nondecreasing spending, a simple strategy for retirees is to invest at least 85% of the wealth in the risk-free asset to set up a *floor portfolio* and the remaining wealth in the stock to set up a *surplus portfolio* with a leverage factor of three. Money is transferred from the surplus portfolio to the floor portfolio, if the value of the surplus portfolio exceeds 15% of the total portfolio value. However, Scott and Watson (2011) overlook the possibility of going bankrupt in the surplus portfolio. To hedge the bankruptcy risk, we propose to insure non-decreasing consumption with put options. Our findings demonstrate that the total costs of buying put options to guarantee nonnegative wealth in the surplus portfolio are fairly low.

We then take into account inflation and compare nominal guarantees with real guarantees. The type of consumption guarantees plays a little role in determining the investment strategy due to the constraint by the Floor-Leverage rule that the floor portfolio can only be invested in riskless asset. However, it has substantial effects on the consumption pattern. The reason is that as inflation erodes future consumption, the retirees with nominal guarantees tend to shift their consumption towards the early periods of retirement.

Finally, we compare the outcomes of the ratchet consumption strategies in terms of

[2]See for example Campbell and Cochrane (1999) and Gomes and Michaelides (2003).

welfare with the classic model of Merton (1969), who considers a continuous-time portfolio and consumption choice model with the time-separable CRRA utility preference. In that model, the fraction of wealth invested in risky assets is constant over time and substantial declines in consumption are possible. We find that the ratchet consumption constraint incurs substantial welfare losses as compared to the optimal strategy in Merton's model. The causes of this efficiency loss are twofold. First, the ratchet consumption model has ineffective smoothing of consumption over time. Second, the ratchet model restricts equity exposure of the retirees in the long run.

The remainder of this chapter is organized as follows. Sections 2.2 and 2.3 review Merton's life-cycle model and the ratchet consumption model, respectively. Section 2.4 introduces the Floor-Leverage rule for ratchet retirees and proposes some variants. Section 2.5 compares the welfare of the various strategies, and section 2.6 concludes with a few policy recommendations.

2.2 Benchmark: Life-Cycle Model with CRRA Utility

This section reviews the life-cycle model with the CRRA utility as the benchmark for the following analysis. This portfolio and consumption choice problem was first analyzed by Samuelson (1969) and Merton (1969). Samuelson (1969) determines the optimal portfolio and consumption strategies for an investor with discrete-time, time-separable utility. Merton (1969) solves the portfolio choice problem in a continuous time setting. For simplicity, we only focus on Merton's continuous-time portfolio choice problem.

In the standard life-cycle model, the expected utility framework is used to describe the preferences of economic agents. Moreover, the utility function takes the form of CRRA (Constant Relative Risk Aversion). Given the time-separable CRRA utility preferences over the consumption, the objective function of an investor with a fixed investment horizon T can be written as

$$\max E\left[\int_0^T e^{-(\alpha+\beta)t}\frac{C_t^{1-\gamma}}{1-\gamma}\mathrm{d}t + e^{-(\alpha+\beta)T}B(W_T)\right], \tag{2.1}$$

where γ is the risk aversion parameter, α is the subjective time discount factor, β is the constant mortality rate, C_t is the consumption at time t, $B(W_T)$ is the bequest function, and W_T is the bequeathed wealth. Equation (2.1) implies that the investor is concerned with maximizing the expected utility from both the consumption streams over her lifetime and her bequeathed wealth. Note that in Merton's model economic agent does not have bequest motive, so that model can be viewed as a special case of Equation (2.1) with $W_T \geq 0$ and $B(W_T) = 0$ for any W_T.

Next, we set up the economy. We assume that there are only two assets available to the investor, a risky asset with constant expected return of μ and volatility of σ and a riskless asset that carries a fixed interest rate of r. Then, the returns of these two assets follow diffusion processes:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}z_t, \tag{2.2}$$

$$\frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t, \tag{2.3}$$

where S and B denote the price of the risky asset and the price of the riskless asset respectively. Then, the portfolio and consumption choice problem for the investor is subject to the budget constraint

$$\mathrm{d}W_t = \left[(x_t(\mu - r) + r)W_t - C_t\right]\mathrm{d}t + x_t W_t \sigma \mathrm{d}z_t \tag{2.4}$$

and the constraints $W_t > 0$ and $C_t > 0$ for $t \in [0, T]$. Here x_t denotes the fraction of wealth invested in the risky asset at time t. Solving this portfolio choice problem using dynamic programming approach yields

$$x_t = \frac{\lambda}{\gamma\sigma}, \tag{2.5}$$

$$C_t = \left[\frac{v}{1 - e^{v(t-T)}}\right] W_t, \tag{2.6}$$

with

$$v = \frac{1}{\gamma}\left[(\alpha + \beta) + (\gamma - 1)\left(\frac{\lambda^2}{2\gamma} + r\right)\right], \tag{2.7}$$

where $\lambda = \frac{\mu - r}{\sigma}$ is the Sharpe ratio. These two equations are the optimal portfolio and consumption strategies respectively. It is evident from Equation (2.5) that the proportion invested in the risky asset is a constant. It is larger, the higher the Sharpe ratio, the lower the volatility, and the lower the risk aversion. Moreover, it is independent of wealth

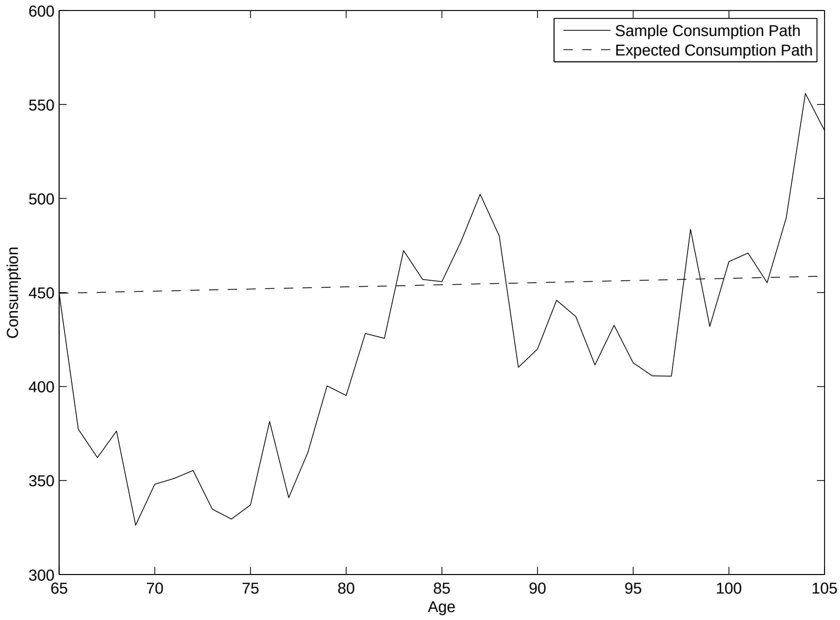


Figure 2.1: Expected and sample consumption paths.

and investment horizon. This is due to the assumption that investment opportunities are time-invariant. As a consequence, the investor becomes myopic and only has a speculative demand in the optimal portfolio.

As for the optimal consumption strategy, Equation (2.6) implies that the fraction of wealth consumed is only time-dependent, but not state-dependent. However, as the wealth level is volatile due to the stock market risk, the consumption level fluctuates over time. For illustrative purpose, we turn to a numerical example. We consider a retiree with age of 65, risk aversion of $\gamma = 3.5$, a 40-year horizon, a subjective discount factor of $\alpha = 0.025$, a constant mortality rate of $\beta = 0.025$[3], and initial wealth of €$10,000$. The riskfree rate and equity risk premium are set to 2% and 4% respectively. The volatility of the stock is 18%. Therefore, by (2.5) we can determine the constant proportion of wealth invested in the risky asset $x = 35.3\%$. As shown in Figure 2.1, given this set of parameter values, the expected consumption stays stable over time. In contrast, the sample consumption exhibits considerable fluctuation and declines in some periods. The retiree will suffer substantially from these intermittent drops in the consumption, if, as the habit formation literature suggests, she regards a large part of their previous consumption as necessary for subsistence and derive utility only from the

[3]As we assume a maximum lifespan of forty years and a constant mortality rate, it immediately follows that the mortality rate $\beta = 1/40 = 0.025$.

excess of consumption above the subsistence level. Therefore, the optimal consumption strategy derived from Merton's portfolio problem does not fit the need of investors with habit persistence. To this end, the following sections are devoted to the analysis of alternative models with habit formation.

2.3 Portfolio Choice with Ratchet Consumption

The ratchet consumption preferences are similar to Merton's assumptions, but in addition require nondecreasing consumption over time. Dybvig (1995) first introduces ratchet consumption preferences into the portfolio choice problem with an infinite investment horizon and finds that the optimal investment strategy is to invest part of the wealth in a risk-free asset to guarantee future spending and the remainder in a leveraged portfolio to seek future increases. Watson and Scott (2011) analyze a similar problem with finite horizon in a discrete time setting.

Watson and Scott (2011) assume a standard Black-Scholes world: there are only two assets traded on the market and their returns follow (2.2) and (2.3). Therefore, the market is complete and there exists a unique pricing kernel. Using the martingale representation approach (Cox and Huang (1989)), the dynamic portfolio choice problem can be mapped into the following static problem:

$$\max_{C_t} \sum_{t=0}^{T} E\left[e^{-(\alpha+\beta)t}\frac{C_t^{1-\gamma}}{1-\gamma}\right], \tag{2.8}$$

$$\text{s.t.} \sum_{t=0}^{T} E\left[M_t C_t\right] \leq W_0, \tag{2.9}$$

$$0 \leq C_0 \leq C_1 \leq \ldots \leq C_T, \tag{2.10}$$

where T is the investment horizon, α is the subjective time discount factor, β is the constant mortality rate and M_t is the pricing kernel. Retirees are assumed to have no intermediate income and no bequest motive. Equation (2.9) is the static budget constraint and Equation (2.10) are the constraints that accommodate investors' need for nondecreasing spending. Watson and Scott (2011) verify that the solution takes the form

$$Y_t = I\left(\frac{\theta M_t}{\mathrm{e}^{-(\alpha+\beta)t} y_t}\right), \tag{2.11}$$

$$C_t = \max(C_{t-1}, Y_t), \tag{2.12}$$

$$\kappa_t = \mathrm{e}^{-(\alpha+\beta)t} U'\left(C_{t-1}\right) \max\left(0, h_t\left[\frac{y_t U'\left(Y_t\right)}{U'(C_{t-1})}\right]\right), \tag{2.13}$$

where θ is the Lagrange multiplier associated with the budget constraint and κs are the multipliers associated with the ratchet consumption constraints. I is the inverse function of the first order derivative of the utility function and Y_t is the consumption for a Merton-Samuelson investor with the time preference function $e^{-(\alpha+\beta)t} y_t$. The functions $h_t(y)$ couple today's consumption decision to expected future decisions and are called coupling functions. The parameters y_t are the zeros of the coupling functions. The last coupling function $h_{T-1}(y) = y - 1$ has the zero $y_{T-1} = 1$. The remaining coupling functions are defined recursively as follows:

$$h_t(y) = y - 1 + \mathrm{e}^{-(\alpha+\beta)} \mathrm{E}_t\left[\max\left(0, h_{t+1}\left(y \mathrm{e}^{-(\alpha+\beta)} \frac{M_{t+1}}{M_t}\right)\right)\right]. \tag{2.14}$$

Equations (2.11) and (2.12) imply that a ratchet investor's optimal consumption at time t (C_t) depends on both his previous period's consumption (C_{t-1}) and his expected future consumption (Y_t) and is a derivative security on the pricing kernel. The derivative's value V_t is a function of three independent variables: the pricing kernel M, the current consumption C, and time t. For any $t' > t$, $V_t\left(M, C, t'\right) = 0$, but at expiration $t' = t$, $V_t\left(M, C, t'\right) = C_t$. At all consumption times $t' \leq t$, C must be updated if there is a new maximum. Hence, V_t is given by

$$V_t\left(M, C, t'^{-}\right) = V_t\left(M, \max\left[C, I\left(\frac{\theta M}{\mathrm{e}^{-(\alpha+\beta)t'} t'}\right)\right], t'\right), \tag{2.15}$$

where the superscript minus on t'^{-} represents an instant prior to t'. Further, the value of the derivative portfolio that replicates an investor's optimal consumption is given by

$$V(M, C, t) = \sum_{n=0}^{T-1-t} V_{t+n}(M, C, t). \tag{2.16}$$

By delta-hedge formula, the fraction of total portfolio wealth invested in the pricing kernel is,

$$f(M, C, t) = \frac{\partial \ln V(M, C, t)}{\partial \ln M}. \tag{2.17}$$

To determine $f(M, C, t)$, one needs to first compute $V(M, C, t)$, which can be done numerically. Then, the fraction of total wealth invested in the risky asset follows from the chain rule,

$$s(M, C, t) = \frac{\partial \ln V(M, C, t)}{\partial \ln M} \frac{\partial \ln M}{\partial \ln S} = \left(-\frac{\mu - r}{\sigma^2}\right) f(M, C, t), \tag{2.18}$$

and the fraction of total wealth invested in the risk-free asset is given by,

$$F(M, C, t) = 1 - s(M, C, t). \tag{2.19}$$

Watson and Scott (2011) claim that a ratchet consumer's optimal investment portfolio can be partitioned into a *floor portfolio* and a *surplus portfolio*. The former invests in the risk-free asset to secure spending at the current level, while the latter invests in risky assets to garner future consumption increases. The consumption level is determined annually in the following way. At the beginning of each year, the retiree first calculates the amount of money D_t needed to sustain €1 of spending throughout the remaining retirement years. Note that D_t is the total price of a ladder of riskless zero-coupon bonds that pay €1 at time t to $T-1$. Therefore, D_t is given by

$$D_t = \sum_{i=t}^{T-1} \mathrm{e}^{-r(i-t)}, \tag{2.20}$$

where r is the risk free interest rate and T is the investment horizon. Second, the retiree needs to determine the minimum floor ratio F_t, which is the minimum fraction of wealth that must be dedicated to sustaining future consumption. The value of F_t is obtained from the optimization solution described above ($F(M, C, t)$). Armed with D_t and F_t, the retiree can calculate C_t, the optimal spending for year t,

$$C_t = \max(C_{t-1}, F_t W_t / D_t), \tag{2.21}$$

where C_{t-1} is the consumption in the previous year and W_t is the current wealth. In short, each year one compares the minimum spending implied from the previous year with the spending sustained by investing $F_t W_t$ in riskless bonds and then chooses the larger one. The initial period consumption is given simply by $C_0 = F_0 W_0 / D_0$.

So far, we have assumed that the inflation rate was zero or, alternatively, that all variables were in real, inflation-adjusted terms. In reality, many pension schemes give only nominal guarantees. Let π denote the constant inflation rate. With the introduction of inflation, Equation (2.10), which captures the ratchet consumption constraints, can be rewritten in nominal terms as,

$$0 \leq C_0 \leq e^{\pi} C_1 \leq \ldots \leq e^{T\pi} C_T, \tag{2.22}$$

where the inflation parameter π controls the maximum rate that real spending C_t is allowed to decrease. If π is zero, inflation is not considered and real consumption never declines. Conversely, if π is greater than zero, nominal spending never declines, but real spending may. The total price of the nominal zero-coupon bonds is

$$\widetilde{D}_t = \sum_{i=t}^{T-1} \mathrm{e}^{-(r+\pi)(i-t)}, \tag{2.23}$$

where r is the real interest rate and the nominal interest rate is the sum of the real rate and inflation $(r + \pi)$. The optimal consumption policy is

$$C_t = \max(e^{-\pi} C_{t-1}, \widetilde{F}_t W_t / \widetilde{D}_t), \tag{2.24}$$

where W_t is the current real wealth and $\widetilde{F}_t$ is obtained from the optimization solution.

2.4 The Floor-Leverage Rule

Scott and Watson (2011) propose a rule of thumb to approximate the complex ratchet consumption policy—the *Floor-Leverage* rule. To guarantee nondecreasing spending in the retirement years, a simple strategy for retirees is to initially allocate 85% of the retirement wealth to the floor portfolio ($F_0 = 85\%$) and all remaining wealth to the surplus portfolio with a leverage factor of three. If the stock market rises, money is transferred from the surplus portfolio to the floor portfolio to maintain a higher level of consumption. They assert that even if the stock market falls, spending is sustained and losses are limited to the surplus portfolio. Nonetheless, given the leveraged position, it is natural to doubt whether the Floor-Leverage rule guarantees non-decreasing ratchet consumption under any circumstances.

To simplify analysis, we assume throughout this section that there is no inflation except for subsection 2.4.4, where the case of sustainable nominal consumption is discussed and compared with that of sustainable real consumption. To begin with, we test the validity of the Floor-Leverage rule and propose some variants. A simulation experiment reveals that there is a positive probability that the value of the surplus portfolio falls below zero, which implies that the transfer of money has to be reversed to keep the surplus portfolio solvent. As a consequence, future consumption is reduced and the Floor-Leverage rule does not guarantee nondecreasing consumption. To remedy this problem, we propose a dynamic trading strategy with put options: to hedge against the downside risk of the stock market, we purchase a series of put options and determine both the strike prices of the put options and put option holdings dynamically. Our findings demonstrate that in the Black-Scholes world the total costs of buying put options account for only a very small fraction of the initial wealth because the strike prices are set at such low levels that only zero value of the surplus portfolios is guaranteed.

In addition, we investigate the dynamic portfolio strategies for different leverage factors and equity premium and identify that future spending increases with the leverage factor due to the higher expected return by taking higher equity exposure. Moreover, comparing sustainable nominal spending with sustainable real spending indicates that the retirees with nominal constraint receive higher consumption stream in the early periods, but have lower consumption growth than their counterparts.

2.4.1 Bankruptcy Risk and Leverage

Following the Floor-Leverage rule proposed in Scott and Watson (2011), the design of the experiment is as follows. First of all, we invest 85% of available assets to purchase a spending floor ($F_0 = 85\%$). Once 85% of the initial wealth is allocated to the floor, the remaining wealth is invested aggressively in equity with a leverage factor of three. After this initial allocation, we check annually whether the surplus portfolio exceeds 15% of the total portfolio. If so, then surplus assets in excess of 15% are reallocated to purchase additional floor spending. Otherwise, no money is transferred from the surplus portfolio to the floor portfolio and consumption level remains the same as the previous period.

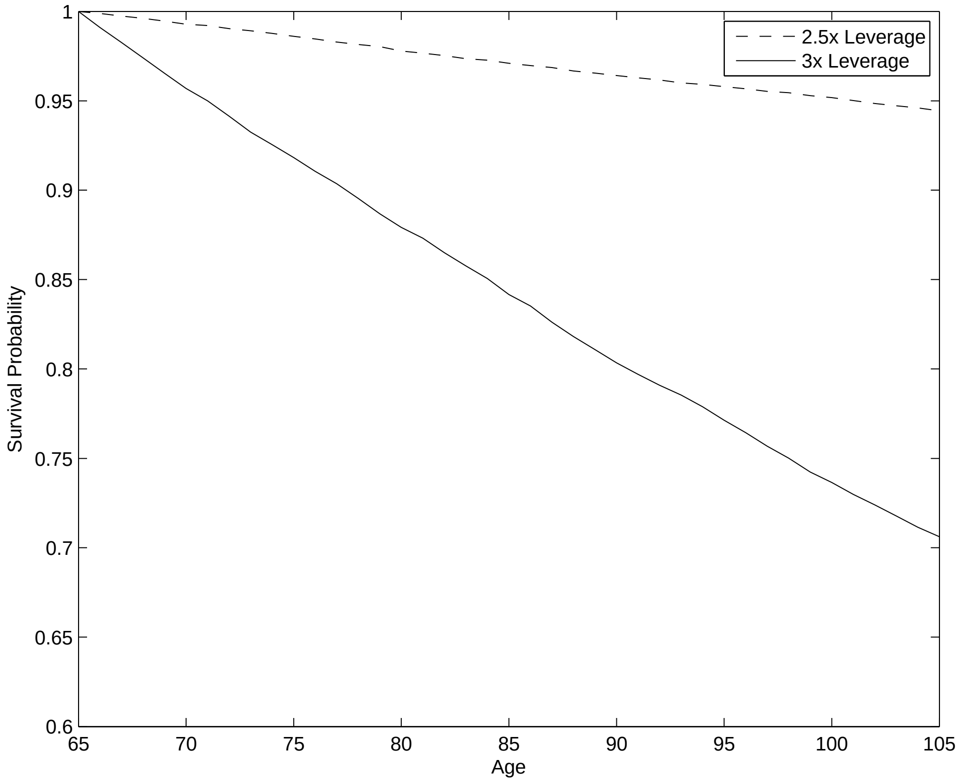


Figure 2.2: Survival probability for different leverage factors.

Moreover, we annually rebalance the surplus portfolio to maintain a constant leverage factor of three. The age of the retiree is 65 at the beginning. Also, we assume that the investment horizon is 40 years ($T = 40$). At each annual review, scenarios, which have negative surplus portfolio value and therefore fail to guarantee nondecreasing future consumption, are eliminated and not be considered in the subsequent periods.

Specifically, we generate $10,000$ scenarios with equal initial wealth of €$10,000$. We investigate how many scenarios survive in each period and how this survivorship evolves over time. The parameter values are the following. We consider a two-asset economy with a riskless interest rate equal to 2% and a risky asset broadly consistent with developed equity markets: an annual risk premium of 4% with an annual volatility of 18%. $10,000$ paths of stock prices are simulated over 20 years with initial stock prices of €100. It is assumed that the leverage is taken by borrowing money at the cost of the real rate and all assets are infinitely divisible and there is no inflation.

Figure 2.2 shows that the survival probability[4] declines almost linearly over time and

[4]Survival probability in each period is calculated as the ratio of the number of the scenarios alive in

reaches a level of 71% at the horizon. This outcome contradicts the argument in Scott and Watson (2011) that the Floor-Leverage rule can ensure nondecreasing consumption over time. In fact, as time goes on, in more and more scenarios the retiree runs out of money in her surplus portfolio because of the occurrence of market crashes. In contrast, when the leverage factor is reduced to 2.5, the survival probability remains above 90% throughout, although it still decreases. Thus, reducing the leverage factors remarkably increases the chance of keeping consumption nondecreasing over time. Nonetheless, as long as there exits a leveraged position, the surplus portfolio is always likely to go bankrupt in extremely bad states of the world, thereby invalidating the argument that the Floor-Leverage rule is able to always generate nondecreasing spending.

2.4.2 Insurance with Put Options

One straightforward strategy to overcome the bankruptcy possibility of the Floor-Leverage rule is to buy a series of put options to hedge against the downside risk of the stock market. The trading strategy is dynamic, because at each annual review the put option holdings must be adjusted in order to obtain a full insurance against bankruptcy risk. We assume a standard Black-Scholes world with complete market, so the prices of the put options can easily be calculated using Black-Scholes option pricing formula. Specifically, in each period, we determine the number of shares (N_t^S), the number of options (N_t^P) and the strike prices of put options (K_t) by solving the following system of equations

$$\begin{cases} N_t^S = N_t^P \\ W_t^{surp} = S_t N_t^S + P_t N_t^P \\ K_t N_t^S = (\frac{L-1}{L} W_t^{surp})\mathrm{e}^r, \end{cases} \tag{2.25}$$

where S_t and P_t denote the prices of the stock and the put option at time t and W_t^{surp}, r and L are the wealth in the surplus portfolio at time t, the borrowing rate and the leverage factor respectively. Note that $L \geqslant 1$ and in the Floor-Leverage rule $L = 3$. The first equation implies that to fully hedge the stock market risk, the number of the put option must be equal to the number of the stock. In the second equation, we calculate the value of the surplus portfolio as the sum of the values of each asset class in the surplus portfolio. Finally, we determine the strike price of the put options such that

that period to the total number of the scenarios generated at the beginning.

Table 2.1: Value of put options as fraction of initial wealth for different volatility

L	$\sigma = 14\%$	$\sigma = 16\%$	$\sigma = 18\%$	$\sigma = 21\%$	$\sigma = 24\%$	$\sigma = 27\%$
3	0.026%	0.086%	0.20%	0.48%	0.90%	1.45%
2.5	0.0014%	0.0085%	0.030%	0.12%	0.28%	0.53%
1	0	0	0	0	0	0
0	0	0	0	0	0	0

This table reports the value of the put options as fraction of initial wealth for different volatility of the stock. L and σ are the leverage factor and the volatility of the stock respectively.

the insured value of the surplus portfolio coincides with the sum of the principal and the interest of the leveraged position, which means that the liquidation value of the surplus portfolio can exactly cover the loan when stock market plunges. Note that as L, r, W_t^{surp} and S_t are known in advance and P_t is a function of K_t, we end up with three equations and three unknowns (N_t^S, N_t^P, and K_t). Due to the complexity of the Black-Scholes formula for P_t, this equation system has to be solved numerically. It can be easily verified that when $W_t^{surp} = 0$, $N_t^S = N_t^P = 0$ and K_t can be any positive real number. Otherwise, there exists a unique solution. The proof is given in Appendix 2.7 .

To show how (2.25) works, we present a numerical example in the following. Suppose we are in the initial period and have €10,000 on hand. By the Floor-Leverage rule, we first invest €8,500 in the riskfree bond to set up the floor portfolio, which ensures a spending level of €305.65 in every future period. Then, we borrow €3,000 to maintain a leverage factor of three and put all the money in equity. As a result, the value of the surplus portfolio is €4,500 and the leveraged position is two thirds of it (€3,000). The initial stock price is assumed to be €100. Plugging these quantities into (2.25), we obtain

$$\begin{cases} N_1^S = N_1^P \\ 4500 = 100N_1^S + P_1 N_1^P \\ K_1 N_1^S = 3060.6. \end{cases} \tag{2.26}$$

Solving for N_1^S, N_1^P and K_1 numerically yields $N_1^S = N_1^P = 44.97$ and $K_1 = 68.04$, which means that in the initial period, the retiree should buy 44.97 stocks and 44.97 put options with the strike price of €68.04, which implies a put option price of €0.06.

Table 2.1 shows the value of put options as a fraction of the initial wealth, which is calculated using the pricing kernel based on simulations.[5] As the retiree starts with

[5]The calculation procedure is as follows. We first use the pricing kernel as the discount factor to

equal endowment of €$10,000$ in all cases, one can easily translate the cost of put options into euro terms. For example, when $L = 3$ and $\sigma = 18\%$, the retiree only needs to pay €20 (€$10,000 \times 0.20\%$) on average for the option insurance. Unsurprisingly, the costs of buying the put options increase with the volatility of the stock. Moreover, the retiree with leveraged factor of 2.5 pays less for the option insurance than her counterpart with leverage of 3, since the former has lower equity exposure and higher survival probability. However, the fraction of wealth allocated to the put options stays small across different strategies[6] and different volatility. There are two reasons. First, the strike prices are set at such low levels that only zero value of the surplus portfolios is guaranteed, thereby generating rather low prices for the put options in the Black-Scholes model. Second, the average amount invested in risky asset is shrinking over time because of the one-way cash flow from the surplus portfolio to the floor portfolio. As a result, there is not much money to insure in many scenarios. It is important to note that there are some difficulties in implementing the option insurance strategy in practice. First, deep out-of-money (OTM) options are lack of liquidity and potentially have large counterparty credit risk. Second, deep OTM options are much more expensive in practice than the Black-Scholes prices, which is known as volatility smile.

2.4.3 Consumption Patterns and Leverage

A vast literature is available on the consumption during retirement. Some studies focus on the behavior of consumption as households transition to retirement and analyze the so-called "retirement consumption puzzle", an abrupt decline in expenditures at retirement.[7] Other papers examine consumption over the life-cycle and provide empirical evidence that the consumption of the retirees decreases over the retirement periods, which seems difficult to reconcile with ratchet consumption preferences.[8] In contrast,

compute the present values of all the put options we purchase. Then, we take the mean of the present values of the put options in a certain period as the expected value of the put options in that period. Finally, we sum up the expected value of the put options in each period to obtain the total expected value of the put options.

[6]Retirees with leverage factor of one borrow no money and only invest the wealth in the surplus portfolio in the stock, while retirees with leverage factor of zero invest all of their wealth in the riskless bond.

[7]See, for example, Aguiar and Hurst (2005), Hurst (2007), Ameriks, Caplin, and Leahy (2007), Hurd and Rohwedder (2003).

[8]See Fernandez-Villaverde and Krueger (2011), Bullard and Feigenbaum (2007), Fernández-Villaverde and Krueger (2007), and Gourinchas and Parker (2002).

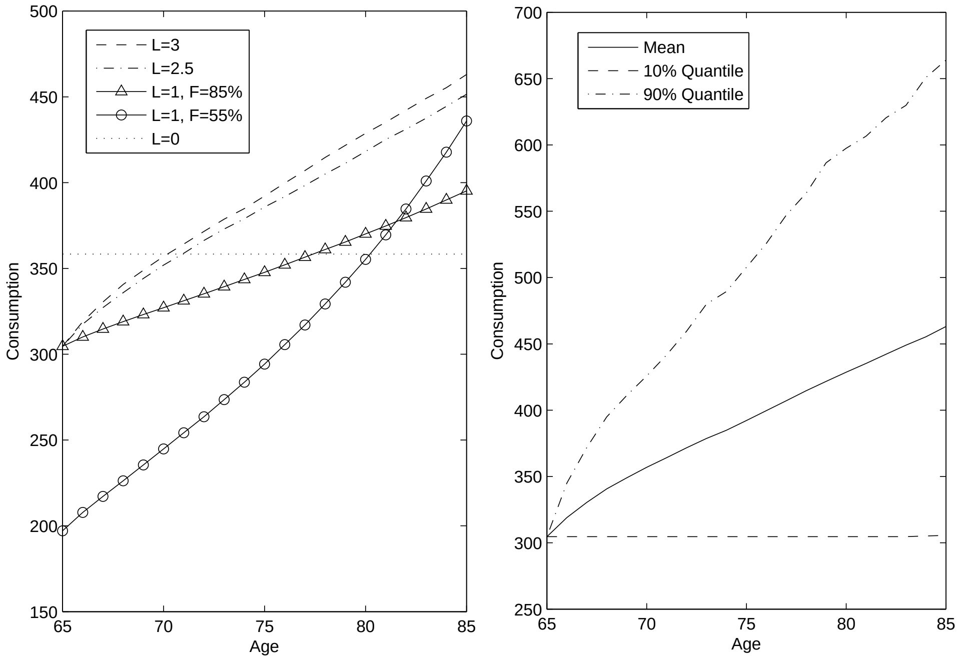


Figure 2.3: Expected consumption using different strategies and confidence bounds with leverage factor of three. The left panel plots the expected consumption using different strategies, while the right panel illustrates confidence bounds with leverage factor of three. L is the leverage factor. F is the floor ratio.

using an internet survey conducted in the U.S. and the Netherlands, Binswanger and Schunk (2011) find that individuals aim to achieve a retirement spending exceeding 70 percent of working life spending and do not want to fall below a certain lower limit of old age spending in both countries, providing evidence in favor of habit persistence. In this subsection, we investigate the patterns of consumption after retirement when the individuals follow the floor-leverage rule for consumption and investments.

Figure 2.3 illustrates the expected consumption of retirees using different leverage strategies.[9] Again, we look at the leveraged strategies with $L = 3$ and $L = 2.5$, with additional put options to prevent bankruptcy in the investment portfolio. Besides the insurance with put options, another simple strategy to get rid of bankruptcy risk is to take no leverage and only use the cash on hand to invest ($L \leq 1$). For a leverage factor of one, we analyze two different strategies. One follows the original Floor-Leverage rule and puts 85% of the wealth in the floor portfolio and the other one changes the floor

[9]We present the results only for the first twenty years, because the expected consumption associated with the strategy ($L = 1, F = 55\%$) in the late periods is so high that the differences between different strategies in early periods become almost invisible.

ratio to 55% so that the initial stock investment is €4500, which coincides with the initial stockholding of the strategy with leverage factor of three.

Figure 2.3 (left panel) illustrates that the retirees with stock investment have an increasing consumption pattern over time, whereas the retirees without stock investment have a constant consumption level. This is because the former types of investors benefit from the positive equity premium and pursue nondecreasing consumption pattern. However, since the retirees without equity exposure don't have the surplus portfolio and use all their wealth to set up the floor portfolio, they consume more than other types of retirees in the early periods. As the leverage factor rises, the slope of consumption curve steepens, which implies that investment strategies with higher leverage factor generate higher expected future spending for retirees.

Since the retirees with leverage factor of three and the retirees with leverage factor of one and floor ratio of 55% have identical initial stock investment, the distinction in the shape of their expected consumption curves reflects the effect of taking leverage. The retirees without leverage have much lower initial consumption than their counterparts, because the only way for them to raise fund for larger equity investment is to cut current consumption. On the other hand, the retirees without leverage enjoy higher consumption growth than the retirees with leverage, both because the latter type of retirees get decreasing benefits from the equity premium due to the reduction of survival probability and because they have to pay for the put options.

The right panel of Figure 2.3 shows the dispersion in consumption for the $L = 3$ case. Consumption at the 90% quantile increases rapidly over time, while the consumption at the 10% quantile remains almost constant. The distinction follows from the market downturns. When the stock prices go down, the surplus portfolio shrinks and becomes financially incapable of raising consumption level. Under extremely bad market conditions, the wealth invested in the surplus portfolio may even decline to zero, leaving consumption constant over the remaining periods and financed completely by the floor portfolio.

2.4.4 Nominal Consumption Guarantees

As many pension plans only provide guarantees for nominal benefit, it is of interest to consider the case with the requirement for nondecreasing nominal consumption. In this

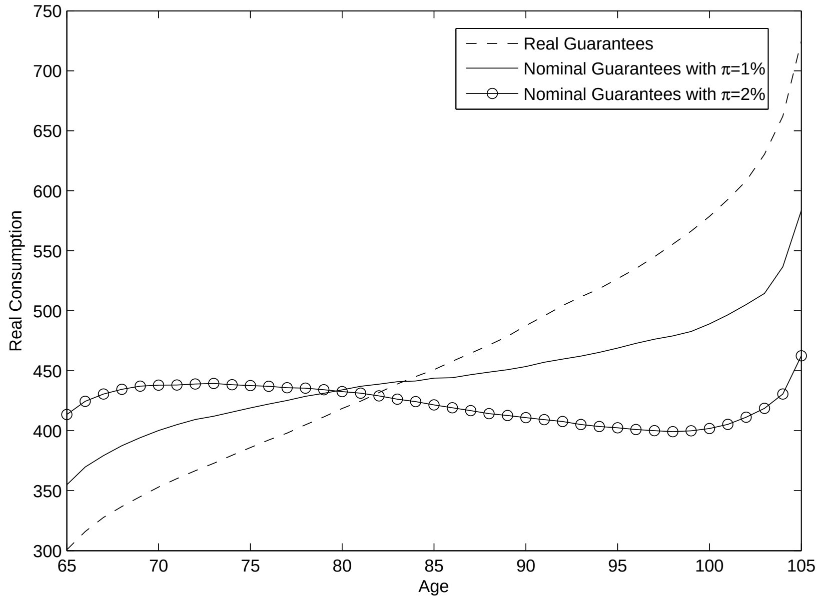


Figure 2.4: Expected real consumption with different types of consumption guarantees and inflation levels.

subsection, we therefore relax the assumption of no inflation, set the inflation rate π equal to some constant levels and raise the stock return by the same amount to keep the equity premium the same as the real guarantee case. In the meantime, other parameter values are held unchanged.

Figure 2.4 illustrates the expected real consumption with different types of consumption guarantees and inflation rates. Obviously, the type of consumption constraint has substantial influence on the expected consumption behavior of the retirees: those requiring nominal guarantees enjoy higher real spending in the early periods but have lower spending growth than their counterparts. Equation (2.23) suggests the total price of the nominal zero-coupon bonds $\widetilde{D}$ decreases with π. The intuition is that increase in the inflation rate raises discount rates on the future nominal consumption (nominal interest rate) and lowers the current price of nominal zero-coupon bonds. Therefore, the shift from the real constraint to the nominal one leads to lower value of D and higher initial consumption level. However, in the nominal guarantee case, the real consumption can be eroded by the inflation in the future periods, thus reducing the real spending growth. The lower consumption growth follows from the fact that the type of consumption guarantees plays very a limited role in determining the equity exposure of the surplus portfolio, as there is no money transfer from the floor portfolio to the surplus portfolio. Hence, the switch between the two types of guarantees only generates a tradeoff between the real spending in the short run and in the long run. On the other hand, for nominal guarantees, the higher the inflation rate, the higher the initial consumption, but the lower the consumption growth. When the inflation rate is set to 2%, the expected real consumption for retirees with nominal guarantees even declines over the late retirement years.

2.5 Welfare Analysis

To examine quantitatively how the leverage factor and equity premium affect the welfare of retirees, we compare the efficiency of different strategies. A welfare criterion is needed for this purpose. Within an expected utility framework, a straightforward method of scoring different strategies goes as follows. We use the optimal dynamic investment strategy in Merton's model as the benchmark[10] and first calculate the utility achieved by adopting this strategy with a given initial wealth (€10,000 in our example). Following Scott and Watson (2011), we model the utility of a spending sequence as the weighted sum of the single year utility—a time-separable model with CRRA utility function. Next, we compute how much cheaper or more expensive we can attain the same utility as benchmark strategy. The result is referred to as efficiency index, which can be used to compare the efficiency of different strategies. Specifically, we consider two benchmark models: one is the optimal strategy in Merton's model and the other one is the optimal strategy in the discrete ratchet consumption model derived by Watson and Scott (2011).

Table 2.2 reports the efficiency analysis of different consumption guarantees and leverage levels.[11] As shown in Panel (a), in the presence of equity investment, the efficiency index increases with the leverage factor given the floor ratio of 85%, which is consistent with the consumption behaviors of different agents in Figure 2.3. However, the efficiency gap declines with the investor's risk aversion. In unreported results, we find that a decrease in equity risk premium also reduces the efficiency gap. Somewhat surprisingly, the pure bond investment strategy is superior to the strategy (L=1, F=0.55) but inferior to other strategies.

[10]To ensure the comparability of different strategies, we assume that the retirees in Merton's model have a finite horizon of 40 years, which is identical to the ratchet retirees' investment horizon.

[11]As the optimal strategy in Merton's model is used as the benchmark, its efficiency is 100% in all cases.

Table 2.2: Efficiency analysis of different strategies using Merton's strategy as benchmark

(a) Real guarantees with $\pi = 0$ and different γ

γ	Merton	L=3	L=2.5	L=1, F=0.85	L=1, F=0.55	L=0
2	100%	75.9%	74.8%	71.2%	63.1%	67.4%
3.5	100%	81.8%	81.3%	77.9%	71.3%	74.2%
5	100%	83.7%	83.3%	81.0%	75.3%	77.2%

(b) Nominal guarantees with $\gamma = 3.5$ and different π

π	Merton	L=3	L=2.5	L=1, F=0.85	L=1, F=0.55	L=0
0	100%	81.8%	81.3%	77.9%	71.3%	74.2%
1%	100%	89.1%	88.6%	85.0%	79.6%	81.4%
2%	100%	94.8%	95.2%	91.3%	85.5%	84.6%

This table reports the efficiency analysis of different strategies using Merton's strategy as the benchmark. In panel (a), inflation is not considered and the guarantees are in real terms. In panel (b), inflation rates vary and the guarantees are in nominal terms, while the risk aversion γ is held constant at 3.5. "Merton" refers to the optimal investment strategy in Merton's model. γ, π and L are the risk aversion, the inflation rate and the leverage factor respectively. The equity premium $(\mu - r)$ is 4%.

The strategy ($L = 1$ and $F = 55\%$) results in considerable welfare losses. This strategy exhibits higher consumption growth, but generates much lower consumption in the initial period and therefore does a very poor job of consumption smoothing over time. Therefore, the strategy with no leverage and a low floor ratio underperforms any other strategies. Furthermore, Merton's strategy dominates all the other strategies. The reasons are twofold. First, in contrast to the Floor-Leverage strategies, it's not constrained from taking large equity exposure in the long run. Second, it generates higher consumption streams in the early periods of the retirement than other strategies, because it does not require substantial saving for nondecreasing future spending.

Panel (b) focuses on nominal consumption guarantees. As the inflation rate rises, the utility loss relative to Merton's strategy shrinks for all the other strategies. This welfare improvement follows from the preference of the retirees towards consumption in the early periods of retirement in the presence of inflation.

Table 2.3 illustrates the efficiency analysis of different strategies using Waston and Scott's strategy as benchmark as the benchmark.[12] The efficiency index remains high across all cases. For example, in case of the floor-leverage rule, the welfare loss for a retiree with $\gamma = 3$ is only 1.9%, which implies that the theoretical optimal strategies are

[12] As the optimal strategy in Waston and Scott's model is used as the benchmark, its efficiency is 100% in all cases.

Table 2.3: Efficiency analysis of different strategies using Waston and Scott's strategy as benchmark

(a) Real guarantees with $\pi = 0$ and different γ

γ	WS	L=3	L=2.5	L=1, F=0.85	L=1, F=0.55	L=0
2	100%	96.7%	95.2%	93.1%	86.1%	90.5%
3.5	100%	98.1%	97.2%	95.9%	91.5%	93.8%
5	100%	98.7%	98.3%	97.4%	94.8%	96.2%

(b) Nominal guarantees with $\gamma = 3.5$ and different π

π	WS	L=3	L=2.5	L=1, F=0.85	L=1, F=0.55	L=0
0	100%	98.1%	97.2%	95.9%	91.5%	93.8%
1%	100%	99.2%	98.9%	96.3%	93.9%	95.7%
2%	100%	99.5%	99.5%	97.4%	94.5%	96.2%

This table reports the efficiency analysis of different strategies using Waston and Scott's strategy as benchmark. In panel (a), inflation is not considered and the guarantees are in real terms. In panel (b), inflation rates vary and the guarantees are in nominal terms, while the risk aversion γ is held constant at 3.5. "WS" refers to the optimal investment strategy in Watson and Scott's model. γ, π and L are the risk aversion, the inflation rate and the leverage factor respectively. The equity premium $(\mu - r)$ is 4%.

well approximated by this simple rue of thumb. Consistent with the results in Table 2.2, the utility cost of implementing the floor-leverage rule is lower for lower risk aversion and higher inflation. Moreover, the welfare losses are much lower than those in the analysis using Merton's model as the benchmark, because the risk taking behavior is severely constrained by the ratchet consumption requirement in Waston and Scott's model.

2.6 Conclusion

In this paper we analyze two different models for consumption after retirement. The first is Merton's rule where consumption is always adjusted to changes in wealth; the second is the so-called ratchet consumption where consumption is guaranteed not to fall over time. Although highly stylized, these rules resemble the benefit rules of the new pension deal in the Netherlands (Merton's rule) and the existing contracts with a nominal floor (ratchet consumption rule).

First, we analyze a simple version of the Ratchet consumption, the Floor-Leverage rule proposed by Scott and Watson (2011). We show that the original Floor-Leverage rule is infeasible and has a high probability of bankruptcy. However, a relatively inexpensive option strategy can hedge against such bankruptcy risk. In contrast, the floor portfolio strategy insures nondecreasing consumption for the retirees. Second, we investigate nominal consumption guarantee and compare it with its real counterpart. The less restrictive nominal guarantees lead to higher initial spending level but lower consumption growth because of the constraint imposed by the Floor-Leverage rule that floor portfolio can only be invested in risk free asset. Third, compared to Merton's consumption rule, the requirement for sustaining previous consumption is very costly in welfare terms. The non-decreasing consumption requires to start with a very low initial consumption, with an expected increasing consumption pattern. This is very costly in welfare terms because of the desire of households to smooth consumption over time.

Based on the previous analysis, we can draw several policy implications for pension funds. First, in terms of investment strategy, if the pension members indicate demand for guarantees, there should be a clear separation of risk-less portfolio and risky portfolio. The reason is that risk-less assets are particularly suitable for ensuring future subsistence consumption, while risky assets are used to increase the return of the overall portfolio and generate consumption growth. Second, real guarantees are very costly from a welfare point of view. Therefore, pension boards should take these costs into account when deciding whether to adopt the new pension contract or stay with the existing one. In contrast, nominal guarantees relax the consumption constraint to a large degree and make much lower welfare losses. Therefore, a replacement with nominal guarantees might be a desirable compromise for the retirees with strong habit persistence.

2.7 Appendix: Proof of Uniqueness of Solution for Equation System (2.25)

First, one can easily transform the equation system into a single equation,

$$P_t(K_t) - \mathrm{e}^{-r}\frac{L}{L-1}K_t + S_t = 0, \tag{2.27}$$

where S_t is the known stock price. Let $f(K_t) = P_t(K_t) - \mathrm{e}^{-r}\frac{L}{L-1}K_t + S_t$. Then, $f(0) = S_t > 0$ and the first derivative of $f(K_t)$ is given by

$$\begin{aligned} f'(K_t) &= P_t'(K_t) - \mathrm{e}^{-r}\frac{L}{L-1} \\ &= \mathrm{e}^{-r(T-t)}N(d_t) - \mathrm{e}^{-r}\frac{L}{L-1}, \end{aligned} \tag{2.28}$$

where $N(d_t)$ is the standard normal cumulative density function and d_t is

$$d_t = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{K_t}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]. \tag{2.29}$$

The second equality in 2.28 follows from the Black-Scholes put option price. As the option portfolio is rebanlanced annually, $T - t \geq 1$ and $\mathrm{e}^{-r(T-t)} < \mathrm{e}^{-r}$. In addition, because $N(d_t) \leq 1 < \frac{L}{L-1}$, $f'(K_t) < 0$, which implies $f(K_t)$ is monotonically decreasing and has only one intersection with x-axis. Therefore, there exists a unique solution for K_t. Once K_t is given, N_t^S and N_t^P can also be uniquely determined. Hence, the solution for the equation system 2.25 is unique.

Chapter 3

Portfolio and Consumption Choice with Habit Formation under Inflation[1]

We investigate the optimal portfolio and consumption policies for a finite-horizon investor in a life-cycle model with habit formation and inflation. We consider two types of habit investors: one forms habit based on past real consumption, while the other on past nominal consumption. The optimal strategy is expressed explicitly in terms of the solution to a linear partial differential equation. We find that the effects of inflation on the optimal strategy are marginal under real habit formation, but substantial under nominal habit formation. Both the hedge portfolio and subsistence portfolio bear much larger inflation exposure in the latter case than in the former and this difference is more pronounced for stronger habit persistence, higher initial habit level and longer investment horizon. We also find that the optimal portfolio is tilted more towards bonds under nominal habit formation than under real habit formation in an incomplete market case of only one nominal bond.

3.1 Introduction

There is empirical evidence that households fear of losing even part of pension benefits and exhibit strong demand for guarantees for their pension income[2]. Habit formation,

[1]This chapter is based on De Jong and Zhou (2013b)

[2]See, for example, Van Rooij, Kool, and Prast (2007) and Antolín, Payet, Whitehouse, and Yermo (2011).

which prescribes that investors form habit on the basis of their own previous consumption and derive utility only from the consumption in excess of the habit levels, provides an explanation for such demand[3]: Pension plan members with habit formation require that their future consumption not fall below a habit consumption level and therefore prefer to receive guarantees in their pension contracts. Moreover, the fact that guaranteed pension payout can be formulated either in real terms or in nominal terms necessitates differentiating the preferences of pension plan members for real guarantees from those for nominal guarantees. In this paper we develop a linear habit formation model that provides a framework to study the optimal portfolio and consumption strategy under different types of guarantees offered by pension funds.

Specifically, we consider two types of habit formation, namely real habit formation and nominal habit formation, which correspond to the demand for real guarantees and nominal guarantees, respectively. Under real habit formation, the real habit level is generated directly by past real consumption rates. In contrast, under nominal habit formation, investors form their nominal habit on the basis of previous nominal consumption but derive utility still from consumption in excess of real value of the habit. This mismatch makes nominal habit formation controversial, however. One explanation for it is money illusion, because investors mistake nominal consumption for real consumption in forming habit levels. There is voluminous literature on money illusion. Although the distinction between nominal and real quantities is obvious, money illusion has proved pervasive as extensively documented in the literature. Shafir, Diamond, and Tversky (1997) establish a psychological foundation for money illusion and propose that people often think of economic transactions in both nominal and real terms and that money illusion arises from an interaction between these representations, which results in a bias toward a nominal evaluation. In the stock market setting, Modigliani and Cohn (1979) postulate that investors suffer from money illusion by discounting future real dividends at nominal rather than real interest rates and this error leads to an inflation-induced mispricing. This is known as "Modigliani-Cohn" hypothesis. Campbell and Vuolteenaho (2004) show in the time series that inflation illusion accounts for a large part of the mispricing in the dividend-price ratio. In the housing market setting, Brunnermeier and Julliard (2008) demonstrate that movements in inflation explain account for a large

[3]To be more precise, this type of habit formation is regarded as internal habit formation. It contrasts with external habit formation where the habit is generated based on the past history of aggregate consumption. In this paper, we narrow our focus to the internal habit formation.

share of the time series variation of the mispricing in the housing market and attribute this finding to money illusion. Given the ample evidence of money illusion in a variety of settings, it is plausible that households likely suffer from money illusion in habit formation and indicate strong demand for nominal guarantees. To this end, despite of the controversy, we still discuss the nominal habit formation case due to the richness of its implications.

This paper introduces inflation to a life-cycle model with habit formation and study the effects of inflation[4] on the optimal portfolio and consumption strategy of a representative habit investor. In particular, we compare both the qualitative and quantitative properties of the optimal portfolio and consumption strategy under different habit formation and link the differences to the different roles of inflation. As the different types of habit formation mimic the demand for different guarantees, our framework can be used to study the optimal portfolio strategy under different forms of pension contracts. We begin by investigating a complete market case as the benchmark and proceed to some incomplete market cases. The analysis of both complete and incomplete market cases is performed under real habit formation and nominal habit formation, respectively. The motivations for the incomplete market case are twofold. First, the perfect hedge against both expected inflation risk and interest rate risk by linear combination of two nominal bonds always requires a short position in one of the bonds, which is unrealistic from a practical point of view. Second, while the inflation-indexed bonds market is well developed in some countries, such as the U.K. and Israel, it is lagging behind in other countries, such as the U.S.. For investors in the latter countries, it is of interest to study the optimal portfolio strategy in the absence of inflation-indexed bonds.

Our main results are as follows: First, consistent with Munk (2008)[5], the optimal portfolio can be explicitly expressed in terms of the solution to a linear partial differential equation and is a combination of three portfolios: (1) a myopic mean-variance portfolio, (2) a hedge portfolio against variation of future investment opportunities in the economy with adjustment of habit formation and (3) a subsistence portfolio ensuring future minimum consumption. Habit formation affects the optimal portfolio strategy through two channels: on the one hand, it induces a subsistence demand and thus reduces the

[4] Note that inflation level and inflation risk play different roles in influencing the optimal consumption and investment decisions: While the real value of a portfolio depends on the inflation level because of inflation erosion, its riskiness is solely affected by the inflation risk.

[5] Munk (2008) study a life-cycle model of consumption and investment with both habit formation and stochastic investment opportunities.

free wealth that can be used for speculative and hedging purposes. This channel is referred to as leverage channel. On the other hand, it adjusts the pricing kernel and alters future investment opportunities.[6] Since the riskiness of future habit levels determines this adjustment, it has a large influence on the hedge portfolio. This channel is referred to as adjustment channel.

Second, the effects of inflation on the optimal portfolio strategy differ considerably between the two cases. Under real habit formation, the importance of inflation is marginal, because it has no influence on the two channels outlined above. On the one hand, since real value of the habit level is not allowed to be eroded by inflation, the leverage effect remains unchanged. On the other hand, the riskiness of future habit levels is solely driven by the uncertainty in the real discount rate. Therefore, the adjustment channel is independent of inflation risk. The mere variation generated by the introduction of inflation is a full hedge against unexpected inflation. In contrast, inflation plays a much bigger role in the case of nominal habit formation. The faster decay of habit shrinks the subsistence portfolio and reduces the leverage effect. This is due to stronger inflation level effect. In the meantime, real value of the habit becomes dependent on inflation risk because of inflation erosion. This change in the riskiness leads to a sharp increase in the inflation risk exposure of the hedge portfolio through the adjustment channel and that of the subsistence portfolio through the leverage channel. Another distinction between the two cases arises with respect to unexpected inflation hedging: the optimal portfolio takes a full insurance against unexpected inflation risk in the case of real habit formation but leaves the subsistence portfolio uninsured in the alternative case because the habit level is permitted to be reduced by inflation and needs no protection.

Third, a comparison with Brennan and Xia (2002)[7] identifies the effects of habit formation on the optimal portfolio in the presence of inflation risk: first of all, a new portfolio for sustaining subsistence consumption shows up. Moreover, the inflation exposure is lower in the case of real habit formation, but higher in the case of nominal habit formation. Third, both the equity exposure and inflation risk exposure become dependent on investment horizon and the horizon effect on the interest rate exposure strengthens.

[6]This is consistent with Schroder and Skiadas (2002), who show that the model with habit formation in a given financial market is closely related to the model without habit formation in a financial market with adjusted price dynamics.

[7]Brennan and Xia (2002) study the optimal portfolio and consumption choice of a finite-horizon investor in the presence of inflation risk.

Fourth, in the incomplete market case of one nominal bond and one inflation-indexed bond, while the optimal portfolio strategy keeps risk exposures are very close to those in the complete market case, it does not take short positions in bonds and can be seen as a good substitute for that in the complete market case. In the case of only one nominal bond, the optimal stock investment, optimal bond investment and stock-to-bond ratio decrease with the habit strength and initial habit level. The horizon effect is negative for the stock investment and positive for the bond investment. Both the optimal stock investment and bond investment are higher under nominal habit formation than those under real habit formation, but the optimal portfolio leans more towards the bond.

Finally, we examine the expected wealth and expected consumption for three types of investors, namely non-habit investor, real habit investor and nominal habit investor. The wealth decumulation is slowest for the nominal habit investor, modest for the real habit investor and fastest for the non-habit investor. The non-habit investor starts with higher consumption but has much lower consumption growth than her counterparts. Within the habit investors, the nominal one has higher wealth and consumption over the whole life-cycle than the real one.

Based on the theoretical analysis, some policy implications can be drawn for long-term investors, particularly for pension funds. First, to ensure future guaranteed pension payout, there should be a clear separation between the subsistence portfolio and other traditional portfolios proposed in the portfolio choice literature. Second, the composition of the hedge portfolio and subsistence portfolio should depend on the type of guarantees offered (real v.s. nominal). Specifically, pension funds offering nominal guarantees should invest more aggressively in stocks and take larger inflation exposure by holding bonds than those offering real guarantees.

This article builds on the strand of papers on dynamic asset allocation with inflation risk. See, for example, Campbell and Viceira (2001), Brennan and Xia (2002), Munk and Sørensen (2004), De Jong (2008), Koijen, Nijman, and Werker (2010) and Van Hemert (2010). In particular, we follow Brennan and Xia (2002) in modeling the asset price dynamics in the presence of inflation risk. On the other hand, this paper also relates to the literature on the optimal portfolio and consumption choice with habit formation in preferences. See, for example, Constantinides (1990), Detemple and Zapatero (1992), Detemple and Karatzas (2003), Bodie, Detemple, Otruba, and Walter (2004) and Munk (2008). Constantinides (1990) derives the optimal portfolio and consumption strategy for an infinitely-lived investor under the assumption of constant investment opportunities. Based on the insightful observation of Schroder and Skiadas (2002) that the model with linear habit formation can be mechanically transformed into an equivalent model without habit formation, Bodie, Detemple, Otruba, and Walter (2004) provide an analysis of optimal portfolio and consumption decision in a more general setting with endogenous labor supply and stochastic wages and Munk (2008) introduces stochastic investment opportunities to the habit-based life-cycle model, which is closest to this paper. We extend Munk's model by allowing inflation risk and study how inflation influences the optimal strategy of the habit investor and how these effects depend on the type of habit formation.

The remainder of the paper is organized as follows. Section 4.2 sets up the model by describing the financial markets and preferences. Section 3.3 presents the solution to the optimization problem. Section 3.4 calibrates the model and carries out some numerical experiments. Section 3.5 concludes the paper and 3.6 shows all proofs.

3.2 The Model

3.2.1 Financial Markets

We follow Brennan and Xia (2002) in modeling the asset price dynamics. There are four variables determining asset prices in the Brennan-Xia model: the nominal stock price S, the instantaneous real interest rate r, the instantaneous expected inflation π and the commodity price level Π. The term structure is characterized by the real interest rate and expected inflation. For simplicity, we assume that the risk premia on sources of uncertainty are constant[8]. The stock price follows a geometric Brownian motion as in the Black and Scholes (1973) model. The real interest rate and expected inflation follow Ornstein-Uhlenbeck processes as in the Vasicek (1977) model. The realized inflation equals the expected inflation plus an i.i.d. unexpected inflation shock. The equations

[8] Koijen, Nijman, and Werker (2010) allows for time variation in risk premia, but abstract from habit formation in preferences.

driving the state variables are given by,

$$\frac{\mathrm{d}S_t}{S_t} = (R_t + \sigma_S\lambda_S)\mathrm{d}t + \sigma_S\mathrm{d}z_{St}, \tag{3.1}$$

$$\mathrm{d}r_t = \kappa_r(\bar{r} - r_t)\mathrm{d}t + \sigma_r\mathrm{d}z_{rt}, \tag{3.2}$$

$$\mathrm{d}\pi_t = \kappa_\pi(\bar{\pi} - \pi_t)\mathrm{d}t + \sigma_\pi\mathrm{d}z_{\pi t}, \tag{3.3}$$

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t\mathrm{d}t + \sigma_\Pi\mathrm{d}z_{\Pi t}, \tag{3.4}$$

where R is the nominal interest rate, λ_S is the nominal price of equity risk, κ_π and κ_r mean reversion parameters, and $\bar{r}$ and $\bar{\pi}$ unconditional means. σ_S, σ_r, σ_π and σ_Π are the volatilities of the stock return, real interest rate, expected inflation and realized inflation, respectively. z_S, z_r, z_π and z_Π are the standard Brownian motions that drive the stock return, real interest rate, expected inflation and realized inflation, respectively. Note that we use uppercase letters for nominal variables and the corresponding lowercase letters for their real counterparts.

We can orthogonalize Equation (3.4) for unexpected inflation:

$$\begin{aligned}\frac{\mathrm{d}\Pi_t}{\Pi_t} &= \pi_t\mathrm{d}t + \xi_S\mathrm{d}z_{St} + \xi_r\mathrm{d}z_{rt} + \xi_\pi\mathrm{d}z_{\pi t} + \xi_u\mathrm{d}z_{ut} \\ &= \pi_t\mathrm{d}t + \xi'\mathrm{d}z_t,\end{aligned} \tag{3.5}$$

where $\mathrm{d}z = (\mathrm{d}z_S, \mathrm{d}z_r, \mathrm{d}z_\pi, \mathrm{d}z_u)'$ denotes the vector of innovations in standard Brownian motions with $\mathrm{d}z_u$[9] orthogonal to $\mathrm{d}z_S$, $\mathrm{d}z_r$, and $\mathrm{d}z_\pi$. The correlation matrix of $\mathrm{d}z$ therefore is

$$\rho = \begin{pmatrix} \{\rho_{S,r,\pi}\}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix}. \tag{3.6}$$

The real pricing kernel of the economy, m_t, follows a diffusion process:

$$\begin{aligned}\frac{\mathrm{d}m_t}{m_t} &= -r_t\mathrm{d}t + \phi_S\mathrm{d}z_{St} + \phi_r\mathrm{d}z_{rt} + \phi_\pi\mathrm{d}z_{\pi t} + \phi_u\mathrm{d}z_{ut} \\ &= -r_t\mathrm{d}t + \phi'\mathrm{d}z_t,\end{aligned} \tag{3.7}$$

where $\phi = (\phi_S, \phi_r, \phi_\pi, \phi_u)'$, represents the constant loadings on the stochastic innovations

[9]Note that in Brennan and Xia (2002), the subscript "u" means unhedgeable. However, as explained below, we consider a complete market in the benchmark model and thus there is no unhedgeable component of inflation risk. We follow this notation for the purpose of comparison.

in the economy and determines the market prices of risk, λ_S, λ_r, λ_π and λ_u, which are associated with innovations $\mathrm{d}z_S$, $\mathrm{d}z_r$, $\mathrm{d}z_\pi$ and $\mathrm{d}z_u$, respectively. Brennan and Xia (2002) show that the nominal short-term risk-free rate R and the vector of nominal market price of risk $\lambda = (\lambda_S, \lambda_r, \lambda_\pi, \lambda_u)'$ are given by

$$\lambda = \rho(\xi - \phi), \tag{3.8}$$

$$R_t = r_t + \pi_t - \xi'\lambda. \tag{3.9}$$

The time t nominal price of a nominal zero-coupon bond maturing at s, denoted by P_t^s, evolves as,

$$\frac{\mathrm{d}P_t^s}{P_t^s} = [R_t - B_r(s-t)\sigma_r\lambda_r - B_\pi(s-t)\sigma_\pi\lambda_\pi]\mathrm{d}t - B_r(s-t)\sigma_r\mathrm{d}z_{rt} - B_\pi(s-t)\sigma_\pi\mathrm{d}z_{\pi t}, \tag{3.10}$$

where

$$B_r(\tau) = \kappa_r^{-1}(1 - e^{-\kappa_r\tau}), \tag{3.11}$$

$$B_\pi(\tau) = \kappa_\pi^{-1}(1 - e^{-\kappa_\pi\tau}). \tag{3.12}$$

In contrast, the time t real price of an inflation-indexed bond maturing at time s evolves as

$$\frac{\mathrm{d}p_t^s}{p_t^s} = [r_t - B_r(s-t)\sigma_r\bar{\lambda}_r]\mathrm{d}t - B_r(s-t)\sigma_r\mathrm{d}z_{rt}, \tag{3.13}$$

where $\bar{\lambda}_r = -\phi'\rho e_2$ and $e_2 = (0,1,0,0)'$. Applying Itô's Lemma to its nominal value, $P_t^{s*} = \Pi_t p_t^s$, yields its nominal return,

$$\frac{\mathrm{d}P_t^{s*}}{P_t^{s*}} = [r_t + \pi_t - B_r(s-t)\sigma_r\lambda_r]\mathrm{d}t - B_r(s-t)\sigma_r\mathrm{d}z_{rt} + \xi'\mathrm{d}z_t. \tag{3.14}$$

Equation (3.10) shows that nominal bonds have loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$, but no loading on $\mathrm{d}z_u$. Thus, in an economy with only stocks and nominal bonds, the inflation process can not be fully spanned and the market is incomplete, which corresponds to the setting of Brennan and Xia (2002). In this chapter, however, we add an inflation-indexed bond to the asset menu to complete the market, because, as shown in (3.14), inflation-indexed bonds have non-zero loading on $\mathrm{d}z_u$, which allows the investor to hedge against unexpected inflation risk. It is important to note that the return processes of nominal bonds of different maturities only differ in their loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$. Hence, any desired combination of loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$ can be achieved by positions in any two bonds of different maturities.

In what follows, we consider both complete market and incomplete market settings. In the complete market case, we assume that the investor can invest in five securities: a nominal riskless asset, a stock, two nominal bonds of different maturities s_1 and s_2 ($s_1 > s_2$) and an inflation-indexed bond of maturity s_3. Let σ be the factor loadings matrix of the stock and three bonds and Λ be the vector of the nominal risk premia, which are given by,

$$\sigma = \begin{pmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & -B_r(s_1)\sigma_r & -B_\pi(s_1)\sigma_\pi & 0 \\ 0 & -B_r(s_2)\sigma_r & -B_\pi(s_2)\sigma_\pi & 0 \\ \xi_S & \xi_r - B_r(s_3)\sigma_r & \xi_\pi & \xi_u \end{pmatrix}, \tag{3.15}$$

and

$$\begin{aligned} \Lambda = \sigma\lambda = (&\sigma_S\lambda_S, -B_r(s_1)\sigma_r\lambda_r - B_\pi(s_1)\sigma_\pi\lambda_\pi, -B_r(s_2)\sigma_r\lambda_r - B_\pi(s_2)\sigma_\pi\lambda_\pi, \\ &-B_r(s_3)\sigma_r\lambda_r + \xi'\lambda)'. \end{aligned} \tag{3.16}$$

In the incomplete market cases, we first exclude the nominal bond with shorter maturity from the asset menu. The motivations for this financial market setting is that the perfect hedge against both expected inflation risk and interest rate risk by linear combination of two nominal bonds always requires a short position in one of the bonds[10]. However, borrowing constraints prevail for most of market participants, making this combination largely infeasible in practice. Then, we consider the setting with the nominal bond of maturity s_1 as the only bond in the market. As for some countries, such as the U.S., the inflation-indexed bond market is relatively underdeveloped, it is of interest to examine the optimal portfolio strategy in the absence of inflation-indexed bonds.

[10]See, for example, Brennan and Xia (2002).

3.2.2 Preferences

We consider an investor with a fixed investment horizon T. The objective of the investor is to maximize over her life-cycle the expected discounted sum of all future utility which are generated by the difference between real consumption c and real habit level h. In line with most of the literature, the utility function is assumed to be of the isoelastic form with risk aversion parameter γ. The individual's portfolio and consumption optimization problem can be formulated as

$$\max_{(C,x)\in A} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{(\frac{C_t}{\Pi_t}-h_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{3.17}$$

where δ is the subjective discount factor, C is the nominal consumption rate, h is the real habit level and A is the set of admissible consumption and portfolio strategy. x is the vector of the portfolio weights on the risky assets and $1 - x'\iota$ is the weight on the nominally riskless asset. The investor maximizes her utility by appropriately choosing a nominal consumption process $C = (C_t)$ and a portfolio strategy $x = (x_t)$. The nominal wealth dynamics can be written as,

$$\mathrm{d}W_t = [W_t(R_t + x_t'\Lambda) - C_t]\,\mathrm{d}t + W_t x_t'\sigma \mathrm{d}z_t. \tag{3.18}$$

The requirement that the future consumption streams must be financeable by the initial wealth of the investor implies a static budget constraint,

$$\mathrm{E}\left[\int_0^T \frac{m_t}{m_0}\frac{C_t}{\Pi_t}\mathrm{d}t\right] \leq \frac{W_0}{\Pi_0}. \tag{3.19}$$

where W_0 is the nominal initial wealth, Π_0 is the initial price level. Choosing C_τ and x_τ over the period $\tau \in [t, T]$ to maximize utility in the remaining lifetime yields the indirect utility:

$$J_t = \max_{(C,x)\in A} \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\delta(s-t)}\frac{(\frac{C_s}{\Pi_s}-h_s)^{1-\gamma}}{1-\gamma}\mathrm{d}s\right]. \tag{3.20}$$

As shown in (3.17), the habit level can be regarded as a subsistence consumption rate, since the consumption rate must exceed the habit level. Note that γ is not the actual level of relative risk aversion, but still an important determinant of it:

$$RRA_t = \gamma \frac{c_t}{c_t - h_t}. \tag{3.21}$$

Obviously, the relative risk aversion is no longer constant, but decreasing in the ratio of consumption to habit. In other words, for any given habit level, higher consumption rate leads to lower risk aversion.

We consider two types of internal habit formation. The first one is real habit formation, in which the real habit level is generated by previous *real* consumption rates,

$$h_t = h_0 \mathrm{e}^{-\beta t} + \alpha \int_0^t \mathrm{e}^{-\beta(t-s)} c_s \mathrm{d}s, \tag{3.22}$$

and evolves as,

$$\mathrm{d}h_t = -(\beta h_t - \alpha c_t)\mathrm{d}t. \tag{3.23}$$

Here c is the real consumption, α is the scaling parameter, β is the persistence parameter and h_0 is the initial real habit level. The real habit level is a weighted average of past consumption rates. The weights are exponentially decreasing so that the recent consumption rates are given higher weights. Following Munk (2008), we require that $\beta > \alpha$ to ensure that the real habit level will decline when her consumption rate coincides with the habit level. Note that when $c_t = h_t$, $\mathrm{d}h_t = -(\beta - \alpha)h_t\mathrm{d}t$. Thus, $(\beta - \alpha)$ can be interpreted as the decay rate of habit level at the minimum consumption and captures habit strength[11].

The alternative is nominal habit formation, in which the nominal habit level is generated by previous *nominal* consumption rates:

$$H_t = H_0 \mathrm{e}^{-\beta t} + \alpha \int_0^t \mathrm{e}^{-\beta(t-s)} C_s \mathrm{d}s. \tag{3.24}$$

As the investor derives utility from consumption on top of real habit level, but forms habit on the basis of previous nominal consumption, there is a mismatch between utility function and habit formation process. This can be considered *money illusion*: the

[11]In what follows, we refer to $(\beta - \alpha)$ as habit strength. But, it is important to note that the smaller $(\beta - \alpha)$, the stronger the habit formation preference.

investor confuses the nominal consumption stream with the real consumption stream in forming habit levels.

Applying Itô's lemma to the relationship $h_t = H_t/\Pi_t$ yields the dynamics of h_t,

$$\mathrm{d}h_t = -\left[\left(\beta + \pi - \xi' \rho \xi\right) h_t + \alpha c_t\right] \mathrm{d}t - h_t \xi' \mathrm{d}z_t. \tag{3.25}$$

A comparison with (3.23) reveals two noteworthy features of real habit dynamics under nominal habit formation: First, the evolution of the real habit level becomes stochastic because of the uncertainty inherited from unexpected inflation. Second, expected inflation enters the drift term, which implies that the real habit level in this case is eroded by inflation and therefore decays faster than that in the case of real habit formation.

3.3 Solutions

3.3.1 Real Habit Formation

Solving the portfolio and consumption optimization problems formulated in Section 3.2 is far from trivial, because linear habit formation produces strong past dependence and renders the utility function not time separable. We follow Schroder and Skiadas (2002) and Munk (2008) in finding the solutions. Schroder and Skiadas (2002) show that the optimal portfolio choice models with habit formation in a given financial markets is closed linked to the corresponding models without habit formation in a financial market with a habit-adjusted price kernel. Applying this relation, Munk (2008) derives a general characterization of the optimal portfolio and consumption strategy and studies the quantitative effects of habit formation in some concrete settings. Under real habit formation, we extend Munk (2008) by incorporating inflation risk and examining how it affects the optimal portfolio strategy in both complete and incomplete market settings.

We first present two auxiliary processes, f and g, which are used to characterize the solutions under real habit formation. The process f is defined by

$$f_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} \frac{m_s}{m_t} \mathrm{d}s \right] = \int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} p_t^s \mathrm{d}s. \tag{3.26}$$

If $c_s = h_s$ for all $s \geq t$, future real habit levels depreciate at a rate of $(\beta - \alpha)$. Hence, f_t

can be thought of as the time t market price of a bond paying continuous real coupons which are declining at the decay rate of real habit levels. This bond can be regarded as habit bond. hf is the cost of ensuring that future real consumption never falls below the current real habit.

The process g is defined by,

$$g_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\delta/\gamma)(s-t)} \left(\frac{m_s}{m_t} \right)^{1-\frac{1}{\gamma}} (1+\alpha f_s)^{1-\frac{1}{\gamma}} \, \mathrm{d}s \right]. \tag{3.27}$$

As $(1+\alpha f)$ can interpreted as the shadow price of one unit of consumption today, g captures the effects of both the habit formation (via f) and the future investment opportunities (via m) on the expected utility. It should be noted that for $\gamma > 1$, both f and g decrease with $(\beta - \alpha)$.

We write the dynamics of f and g as

$$\mathrm{d}f_t = f_t \left[\mu_{ft} \mathrm{d}t + \sigma'_{ft} \mathrm{d}z_t \right], \tag{3.28}$$

$$\mathrm{d}g_t = g_t \left[\mu_{gt} \mathrm{d}t + \sigma'_{gt} \mathrm{d}z_t \right], \tag{3.29}$$

where

$$\sigma_{gt} = \left(0, \frac{\partial g / \partial r(r,t)}{g(r,t)} \sigma_r, 0, 0 \right)', \tag{3.30}$$

$$\sigma_{ft} = \left(0, \frac{-\int_t^T B_r(s-t) \mathrm{e}^{-(\beta-\alpha)(s-t)} p_t^s \mathrm{d}s}{\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} p_t^s \mathrm{d}s} \sigma_r, 0, 0 \right)', \tag{3.31}$$

and μ_f and μ_g are some adapted processes. Equation (3.31) shows that under real habit formation, the volatilities of f and g are driven solely by the interest rate risk. This stems from the fact that real zero-coupon bonds, which constitute f, only carry exposure to interest rate risk and this exposure is passed on to g through f.

Theorem 1 characterizes the optimal strategy in terms of the solution of a one dimensional, second order PDE for g.

Theorem 1. *Assume that $w_0 \geq h_0 f_0$. The indirect utility is*

$$J_t = \frac{{g_t}^{\gamma} (w_t^* - h_t^* f_t)^{1-\gamma}}{1-\gamma} \tag{3.32}$$

and $g(r,t)$ solves the PDE,

$$\begin{aligned}
&\frac{\partial g}{\partial t}(r,t) + \left[\kappa_r(\bar{r} - r_t) + \left(1 - \frac{1}{\gamma}\right)\sigma_r\phi_r\right]\frac{\partial g}{\partial r}(r,t) + \frac{1}{2}\sigma_r^2\frac{\partial^2 g}{\partial r^2}(r,t) \\
&\quad + [1 + \alpha f(r,t)]^{1-\frac{1}{\gamma}} = \left[\frac{\delta}{\gamma} + \left(1 - \frac{1}{\gamma}\right) r_t + \frac{\gamma - 1}{2\gamma^2}\phi'\rho\phi\right] g(r,t)
\end{aligned} \tag{3.33}$$

with the terminal condition $g(r,T) = 0$. The optimal real consumption strategy, c^*, is

$$c_t^* = h_t^* + (1 + \alpha f_t)^{-\frac{1}{\gamma}}\frac{w_t^* - h_t^* f_t}{g_t}. \tag{3.34}$$

The optimal portfolio strategy, $x^* = (x_S^*, x_{N_1}^*, x_{N_2}^*, x_I^*)'$, is

$$\begin{aligned}
x_t^* &= \frac{w_t^* - h_t^* f_t}{w_t^*}\frac{1}{\gamma}(\sigma')^{-1}(-\phi) + \frac{w_t^* - h_t^* f_t}{w_t^*}(\sigma')^{-1}\sigma_{gt} + \frac{h_t^* f_t}{w_t^*}(\sigma')^{-1}\sigma_{ft} + (\sigma')^{-1}\xi \\
&= \frac{w_t^* - h_t^* f_t}{w_t^*}\frac{1}{\gamma}\Sigma^{-1}\Lambda + \frac{w_t^* - h_t^* f_t}{w_t^*}\left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\sigma\rho(\hat{\sigma}_{gt} + \xi) + \frac{h_t^* f_t}{w_t^*}\Sigma^{-1}\sigma\rho(\sigma_{ft} + \xi),
\end{aligned} \tag{3.35}$$

where $\hat{\sigma}_g = \left(\frac{\gamma}{\gamma-1}\right)\sigma_g$, w^* is the real wealth process induced by the optimal strategy, h^* is the real habit level induced by the optimal real consumption strategy and $\Sigma = \sigma\rho\sigma'$ is the variance-covariance matrix of the nominal asset returns.

The condition $w_0 \geq h_0 f_0$ ensures that the initial wealth of the investor can sustain the minimum consumption level in the future. As shown in Appendix 3.6.1, we first derive the solution of the dual model without habit formation, which is closely related to the model of Brennan and Xia (2002) and then transform it to the solution of the primal model with habit formation by applying the results of Schroder and Skiadas (2002) to the case with inflation risk.

The optimal consumption in (3.34) contains two components: the current habit level and a time and state-dependent fraction of the free wealth $w - hf$. Since both f and g decrease with $(\beta - \alpha)$ for $\gamma > 1$, the marginal propensity to consume $(1 + \alpha f)^{-1/\gamma}/g$ and the consumption rate increase with $(\beta - \alpha)$, implying that as the habit strength declines the investor tends to consume more out of her wealth. As f and g have no loadings on both expected and unexpected inflation risk factors, the optimal consumption strategy is unaffected by inflation risk.

Equation (3.35) expresses the optimal portfolio as the sum of three portfolios: a myopic portfolio that invests in the nominal mean-variance tangency portfolio represented by $\Sigma^{-1}\Lambda$, a hedge portfolio that provides hedge against variation of future investment opportunities in the economy modified by the presence of habit formation, and a subsistence portfolio that ensures future minimum consumption. As the presence of habit formation induces the investor to set aside a fraction of wealth for future minimum consumption stream, the free wealth is reduced to $w - hf$, which dampens both the myopic demand and the hedge demand. In addition to this leverage effect, habit formation affects the hedge demand also through σ_g. Equation (3.62) in Appendix 3.6.1 shows that the habit-adjusted pricing kernel, which determines the investment opportunities in the presence of habit formation, involves f. Therefore, the optimal hedge against variations in future investment opportunities must take into account the changes in the cost of ensuring the minimum consumption level.

A comparison with Munk (2008) reveals that under real habit formation, the effects of inflation on the optimal portfolio strategy are very small: it only induces a hedge against unexpected inflation, which corresponds to the term $(\sigma')^{-1}\xi$. This is a direct consequence of no influence of inflation on habit formation: since σ_f and σ_g are unaffected by inflation risk, both the hedge portfolio and the subsistence portfolio carry exposure to inflation risk only through the hedge against unexpected inflation, which is consistent with Brennan and Xia (2002).

Turning to the incomplete market cases, we follow the approach taken in De Jong (2008) to derive an approximate solution to the optimization problem under real habit formation. The rationale behind the approach is to minimize a pre-specified norm of the difference between optimal and feasible wealth dynamics. Theorem 2 characterizes the approximate solution.

Theorem 2. *An approximate solution in the incomplete market, x_t^*, is*

$$\begin{aligned} x_t^* &= \frac{w_t^* - h_t^* f_t}{w_t^*}\frac{1}{\gamma}\Sigma_I^{-1}\sigma_I\rho(-\phi) + \frac{w_t^* - h_t^* f_t}{w_t^*}\Sigma_I^{-1}\sigma_I\rho\sigma_{gt} + \frac{h_t^* f_t}{w_t^*}\Sigma_I^{-1}\sigma_I\rho\sigma_{ft} + \Sigma_I^{-1}\sigma_I\rho\xi \\ &= \frac{w_t^* - h_t^* f_t}{w_t^*}\frac{1}{\gamma}\Sigma_I^{-1}\Lambda_I + \frac{w_t^* - h_t^* f_t}{w_t^*}\left(1 - \frac{1}{\gamma}\right)\Sigma_I^{-1}\sigma_I\rho(\hat{\sigma}_{gt} + \xi) + \frac{h_t^* f_t}{w_t^*}\Sigma_I^{-1}\sigma_I\rho(\sigma_{ft} + \xi), \end{aligned} \tag{3.36}$$

where σ_I is the factor loadings matrix of the risky assets in the complete market, Λ_I is the vector of the nominal risk premia and $\Sigma_I = \sigma_I\rho\sigma_I'$ is the variance-covariance matrix *of the nominal asset returns*[12].

3.3.2 Nominal Habit Formation

In this subsection, we turn to nominal habit persistence, which is formed based on the households' previous nominal consumption. The individual's portfolio and consumption optimization problem can be reformulated as

$$\max_{(C,x)\in A} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{\left(\frac{C_t-H_t}{\Pi_t}\right)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{3.37}$$

where H is the nominal habit level defined by,

$$H_t = H_0\mathrm{e}^{-\beta t} + \alpha\int_0^t \mathrm{e}^{-\beta(t-s)}C_s\mathrm{d}s \tag{3.38}$$

Once again, we present the solution in terms of two auxiliary processes denoted by $\hat{f}$ and $\hat{g}$, respectively. The process $\hat{f}$ is defined by

$$\hat{f}_t = \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)}\frac{m_s/m_t}{\Pi_s/\Pi_t}\mathrm{d}s\right] = \int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)}P_t^s\mathrm{d}s. \tag{3.39}$$

If $C_s = H_s$ for all $s \geq t$, future nominal habit levels depreciate at a rate of $(\beta-\alpha)$. Hence, $\hat{f}$ can be thought of as the time t market price of a bond paying continuous nominal coupons which are declining at the decay rate of nominal habit levels and $H\hat{f}$ is the cost of ensuring that future nominal consumption never falls below the current nominal habit. A comparison between (3.26) and (3.39) shows that the habit bond under nominal habit formation is comprised of nominal zero-coupon bonds rather than inflation-indexed zero-coupon bonds. It is worth noting that under the calibrated parameter values shown below, $f_t > \hat{f}_t$ for any $t < T$, which implies that the habit bond is cheaper under nominal habit formation than under real habit formation. This can be explained by the fact that in the case of nominal habit formation, the real habit level is allowed to be eroded by inflation and depreciates faster. Since the values of future coupons decline, the price of

[12] In the case of one nominal bond and one inflation-indexed bond, $x_t^* = (x_{St}^*, x_{Nt}^*, x_{It}^*)'$, σ_I is a matrix containing the first, second and fourth rows of σ and Λ_I is a vector containing the first, second and fourth rows of Λ. In contrast, in the case of only one nominal bond, $x_t^* = (x_{St}^*, x_{Nt}^*)'$, σ_I is a matrix containing the first and second rows of σ and Λ_I is a vector containing the first and second rows of Λ.

the habit bond drops.

The process $\hat{g}$ is defined by,

$$\hat{g}_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\delta/\gamma)(s-t)} \left(\frac{m_s}{m_t} \right)^{1-\frac{1}{\gamma}} \left(1 + \alpha \hat{f}_s \right)^{1-\frac{1}{\gamma}} \mathrm{d}s \right]. \tag{3.40}$$

$\hat{g}$ captures the effects of both the habit formation (via $\hat{f}$) and the future investment opportunities (via m) on the expected utility. It should be noted that for $\gamma > 1$, both $\hat{f}$ and $\hat{g}$ decrease with $(\beta - \alpha)$ and $\hat{g} < g$.

We define the dynamics of $\hat{f}$ and $\hat{g}$ as

$$\mathrm{d}\hat{f}_t = \hat{f}_t \left[\mu_{\hat{f}t} \mathrm{d}t + \sigma'_{\hat{f}t} \mathrm{d}z_t \right] \tag{3.41}$$

$$\mathrm{d}\hat{g}_t = \hat{g}_t \left[\mu_{\hat{g}t} \mathrm{d}t + \sigma'_{\hat{g}t} \mathrm{d}z_t \right] \tag{3.42}$$

where

$$\sigma_{\hat{g}t} = \left(0, \frac{\partial \hat{g}/\partial r(r,\pi,t)}{\hat{g}(r,\pi,t)} \sigma_r, \frac{\partial \hat{g}/\partial \pi(r,\pi,t)}{\hat{g}(r,\pi,t)} \sigma_\pi, 0 \right)', \tag{3.43}$$

$$\sigma_{\hat{f}t} = \left(0, \frac{-\int_t^T B_r(s-t) \mathrm{e}^{-(\beta-\alpha)(s-t)} P_t^s \mathrm{d}s}{\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} P_t^s \mathrm{d}s} \sigma_r, \frac{-\int_t^T B_\pi(s-t) \mathrm{e}^{-(\beta-\alpha)(s-t)} P_t^s \mathrm{d}s}{\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} P_t^s \mathrm{d}s} \sigma_\pi, 0 \right)'. \tag{3.44}$$

and $\mu_{\hat{f}}$ and $\mu_{\hat{g}}$ are some adapted processes. Equation (3.44) shows that under nominal habit formation, the volatilities of $\hat{f}$ and $\hat{g}$ are driven by both the interest rate risk and expected inflation risk. This is because nominal zero-coupon bonds, which constitute $\hat{f}$, are exposed to both risk factors and $\hat{g}$ inherit these exposures from $\hat{f}$.

Theorem 3 characterizes the optimal strategy in terms of the solution of a two dimensional, second order PDE for $\hat{g}$.

Theorem 3. *Assume that $w_0 \geq h_0 \hat{f}_0$. The indirect utility is*

$$J_t = \frac{\hat{g}_t^\gamma (w_t^* - h_t^* \hat{f}_t)^{1-\gamma}}{1-\gamma} \tag{3.45}$$

and $\hat{g}(r,\pi,t)$ *solves the PDE,*

$$\begin{aligned}\left[\frac{\delta}{\gamma}+\left(1-\frac{1}{\gamma}\right)r_t+\frac{\gamma-1}{2\gamma^2}\phi'\rho\phi\right]\hat{g}(r,\pi,t)&=\frac{\partial\hat{g}}{\partial t}(r,\pi,t)+\left[1+\alpha\hat{f}(r,\pi,t)\right]^{1-\frac{1}{\gamma}}\\&+\left[\kappa_r(\bar{r}-r_t)+\left(1-\frac{1}{\gamma}\right)\sigma_r\phi_r\right]\frac{\partial\hat{g}}{\partial r}(r,\pi,t)+\frac{1}{2}\sigma_r^2\frac{\partial^2\hat{g}}{\partial r^2}(r,\pi,t)\\&+\left[\kappa_\pi(\bar{\pi}-\pi_t)+\left(1-\frac{1}{\gamma}\right)\sigma_\pi\phi_\pi\right]\frac{\partial\hat{g}}{\partial\pi}(r,\pi,t)+\frac{1}{2}\sigma_\pi^2\frac{\partial^2\hat{g}}{\partial\pi^2}(r,\pi,t)\end{aligned}\tag{3.46}$$

with the terminal condition $\hat{g}(r,\pi,T)=0$. *The optimal real consumption strategy is*

$$c_t^*=h_t^*+(1+\alpha\hat{f}_t)^{-\frac{1}{\gamma}}\frac{w_t^*-h_t^*\hat{f}_t}{\hat{g}_t}.\tag{3.47}$$

The optimal portfolio strategy, $x_t^*=(x_{St}^*,x_{N_1t}^*,x_{N_2t}^*,x_{It}^*)'$, *is given by*

$$\begin{aligned}x_t^*&=\frac{W_t^*-H_t^*\hat{f}_t}{W_t^*}\frac{1}{\gamma}(\sigma')^{-1}(-\phi)+\frac{W_t^*-H_t^*\hat{f}_t}{W_t^*}(\sigma')^{-1}\sigma_{\hat{g}t}\\&\quad+\frac{H_t^*\hat{f}_t}{W_t^*}(\sigma')^{-1}\sigma_{\hat{f}t}+\frac{W_t^*-H_t^*\hat{f}_t}{W_t^*}(\sigma')^{-1}\xi\\&=\frac{w_t^*-h_t^*\hat{f}_t}{w_t^*}\frac{1}{\gamma}\Sigma^{-1}\Lambda+\frac{w_t^*-h_t^*\hat{f}_t}{w_t^*}\left(1-\frac{1}{\gamma}\right)\Sigma^{-1}\sigma\rho(\hat{\sigma}_{\hat{g}t}+\xi)+\frac{h_t^*\hat{f}_t}{w^*}\Sigma^{-1}\sigma\rho\sigma_{\hat{f}t},\end{aligned}\tag{3.48}$$

where $\hat{\sigma}_{\hat{g}t}=\left(\frac{\gamma}{\gamma-1}\right)\sigma_{\hat{g}t}$.

We assume that $\gamma>1$ and focus on the comparison between the optimal strategy in two cases. The relation $\hat{f}<f$ implies that the value of the habit bond declines. This is a result of erosion by inflation: as the inflation drives the real habit level to decay faster, the habit bond price goes down and therefore less money is needed to ensure future subsistence consumption. The relation $\hat{g}<g$, together with $\hat{f}<f$ implies that the marginal propensity to consume $(1+\alpha f)^{-1/\gamma}/g$ and the consumption rate increase.

On the other hand, there are some major changes to the optimal portfolio strategy. First, the reduction in the value of the habit bond leads to weaker leverage effects and lower subsistence demand. Therefore, the speculative portfolio expands while the subsistence portfolio shrinks. However, it is not possible to determine analytically how the hedge portfolio changes between the two cases, because habit persistence influences the hedge portfolio not only through the leverage effect but also through its effect on investment opportunities and the latter effect has to be evaluated numerically. Second, the inflation risk has much larger impact on the optimal portfolio than it does under the real habit formation. The explanation for this bigger effect is that the habit bond $\hat{f}$, which determines not only the risk profile of the subsistence portfolio but also future investment opportunities, is comprised of nominal zero-coupon bonds and therefore bears expected inflation risk. As a result, both σ_f and σ_g become subject to expected inflation risk, thereby substantially increasing the inflation risk exposures of both the hedge portfolio and the subsistence portfolio. Third, the optimal portfolio no longer takes full insurance against unexpected inflation risk; the subsistence portfolio is left uninsured. This is because under nominal habit formation the real habit level is permitted to be reduced by inflation and therefore has a perfectly negative correlation with realized inflation, which is clearly shown in (3.25).

Turning to the incomplete cases, we once again follow the approach taken in De Jong (2008) to find an approximate solution under nominal habit formation. Theorem 4 characterizes the solution.

Theorem 4. *An approximate solution in the incomplete market, x_t^*, is*

$$\begin{aligned} x_t^* &= \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \frac{1}{\gamma} \Sigma_I^{-1} \sigma_I \rho (-\phi) + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \sigma_{\hat{g}t} \\ &\quad + \frac{h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \sigma_{\hat{f}t} + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \xi \\ &= \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \frac{1}{\gamma} \Sigma_I^{-1} \Lambda_I + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \left(1 - \frac{1}{\gamma}\right) \Sigma_I^{-1} \sigma_I \rho (\hat{\sigma}_{\hat{g}t} + \xi) + \frac{h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \sigma_{\hat{f}t}. \end{aligned} \tag{3.49}$$

3.4 Numerical Illustrations

In this section, we carry out some numerical experiments to compare the effects of inflation and habit persistence on the optimal consumption and portfolio strategy under different types of habit formation. In the benchmark case, we consider an investor with risk aversion parameter $\gamma = 3$, a 30-year horizon, and a time preference rate $\delta = 0.02$. Initial wealth, initial habit level and initial price level are set to $W_0 = 10000$, $h_0 = 400$ and $\Pi_0 = 1$, respectively. Habit parameters are taken to be $\alpha = 0.3$ and $\beta = 0.4$. To calibrate the model, we follow the parameter estimates reported in Brennan and Xia

Table 3.1: Parameter values

Parameter	Value
Stock return process: $\mathrm{d}S/S = (R_f + \lambda_S\sigma_S)\mathrm{d}t + \sigma_S\mathrm{d}z_S$	
σ_S	0.158
λ_S	0.343
Real interest rate process: $\mathrm{d}r = \kappa_r(\bar{r} - r)\mathrm{d}t + \sigma_r\mathrm{d}z_r$	
$\bar{r}$	0.017
κ_r	0.105
σ_r	0.013
λ_r	-0.209
Expected inflation process: $\mathrm{d}\pi = \kappa_\pi(\bar{\pi} - \pi)\mathrm{d}t + \sigma_\pi\mathrm{d}z_\pi$	
$\bar{\pi}$	0.054
κ_π	0.027
σ_π	0.014
λ_π	-0.105
Realized inflation process: $\frac{\mathrm{d}\Pi}{\Pi} = \pi\mathrm{d}t + \xi_S\mathrm{d}z_S + \xi_r\mathrm{d}z_r + \xi_\pi\mathrm{d}z_\pi + \xi_u\mathrm{d}z_u$	
ξ_S	0
ξ_r	0
$\xi\pi$	0
ξ_u	0.013
Pricing kernel process: $\frac{\mathrm{d}m}{m} = -r\mathrm{d}t + \phi_S\mathrm{d}z_S + \phi_r\mathrm{d}z_r + \phi_\pi\mathrm{d}z_\pi + \phi_u\mathrm{d}z_u$	
ϕ_S	-0.333
ϕ_r	0.170
ϕ_π	0.120
ϕ_u	0
Correlations	
ρ_{Sr}	-0.129
$\rho_{S\pi}$	-0.024
$\rho_{r\pi}$	-0.061

This table shows the parameter values taken from Brennan and Xia (2002).

(2002), which are shown in Table 3.1. Note that we assume that unexpected inflation is uncorrelated with stock returns, real interest rate and expected inflation, so that only inflation-indexed bonds can be used to hedge against unexpected inflation. In the cases with complete market, we assume that the there are three bonds available to the investor, namely an 1-year nominal bond ($s_1 = 1$), an 10-year nominal bond ($s_2 = 10$) and an 10-year inflation-indexed bond ($s_3 = 10$). Results under real habit formation are obtained by solving the one dimensional PDE (3.33) for g using a Crank-Nicolson finite difference scheme, with 500 real interest rate subintervals and 1000 time steps. In contrast, results under nominal habit formation are obtained by solving the two dimensional PDE (3.46) for g using an explicit finite difference scheme, with 50 real interest rate subintervals, 50 expected inflation subintervals and 1000 time steps.

We can calculate the loadings of the optimal portfolio on the innovations in different risk factors to decompose its risk exposure:

$$L_S = x_S + x_I \frac{\xi_S}{\sigma_S}, \tag{3.50}$$

$$L_r = -x_{N1} B_r(s_1) - x_{N2} B_r(s_2) - x_I \left[B_r(s_3) - \frac{\xi_r}{\sigma_r} \right], \tag{3.51}$$

$$L_\pi = -x_{N1} B_\pi(s_1) - x_{N2} B_\pi(s_2) + x_I \frac{\xi_\pi}{\sigma_\pi}, \tag{3.52}$$

$$L_u = x_I. \tag{3.53}$$

Because we assume $\xi_S = 0$, the loadings on the innovations in the equity risk and unexpected inflation risk coincide with the optimal stock allocation and optimal inflation-indexed bond allocation. Hence, in what follows we do not report L_S and L_u. It should be noted that the stock is only contained in the myopic portfolio, because it is appropriate neither for hedging purpose nor for ensuring the future subsistence consumption.

Table 3.2 summarizes the optimal portfolio strategy in complete market under real habit formation. As shown in panel (a), the introduction of habit formation remarkably reduces the equity exposure and expected inflation risk exposure and this effect is more pronounced for stronger habit formation, which is associated with smaller $(\beta-\alpha)$. These lower risk exposures can be attributed to the reduction of the free wealth, because under real habit formation the equity risk and expected inflation risk are only taken by the myopic portfolio. In contrast, habit strength has different effects on the interest risk exposure of different portfolios. While the leverage effect reduces the myopic demand and hedge demand, the expansion of the subsistence demand driven by larger habit strength leads to higher interest rate loadings. Moreover, habit strength can affect the hedge portfolio also by changing the volatility of habit-adjusted investment opportunities. The observation that the interest rate sensitivity is decreasing in habit strength indicates that the leverage effect dominates. The lower interest rate and inflation risk exposures associated with weaker habit persistence reduce the absolute demand for both nominal bonds. Panel (b) shows that as initial habit level rises, the optimal portfolio takes less interest rate risk exposure and inflation risk exposure and reduces the holdings of the stock and the two nominal bonds because of the pure leverage effect.

Panel (c) illustrates the importance of investment horizon. Equity exposure and inflation exposure are decreasing in the investment horizon, since longer horizon substantially increases the price of the habit bond and generates stronger leverage effect. On the contrary, the optimal interest rate loadings rise with investment horizon. The reason is that the volatilities of both the habit bond σ_f and future investment opportunities σ_g increase sharply, which induces much larger subsistence demand and hedge demand and offsets the leverage effect. As a result, the absolute portfolio shares in both nominal bonds are higher for longer horizon. These observations stand in stark contrast to Brennan and Xia (2002), who find limited horizon effect on the optimal interest rate risk exposure (about five years) and no horizon effects on the optimal equity exposure and optimal inflation risk exposure. Finally, the optimal inflation-indexed bond holding is independent of habit parameters and investment horizon, because the optimal portfolio simply takes a full insurance against the unexpected inflation risk, which corresponds to the term $(\sigma')^{-1}\xi$.

Table 3.3 shows the approximately optimal portfolio strategy with one nominal bond and one inflation-indexed bond under real habit formation. Interestingly, the exposures of the portfolio to equity risk, interest rate risk and expected inflation risk in this case are very close to those in the complete market case, suggesting that there is only little deviation from the optimal strategy. In contrast to the results in Table 3.2, the holdings of the two bonds keep constant in all scenarios. The demand for the inflation-indexed bond is mainly driven by the interest rate exposure of the portfolio because it does not load on the expected inflation. On the contrary, the holding of the nominal bond is primarily affected by the expected inflation exposure. As a consequence, the allocations to the nominal and inflation-indexed bonds move in the same direction as the absolute expected inflation exposure and interest rate exposure, respectively.

Table 3.4 reports the approximately optimal portfolio strategy with one nominal bond under real habit formation. From panel (a) we can see that the presence of habit persistence in preference drives down the demand for both risky assets because of the leverage effect. While both the stock holding and bond holding decrease with habit strength, the whole portfolio tilts towards the bond. This is a result of higher hedge demand and subsistence demand induced by stronger habit persistence. Panel (b) shows that higher initial habit level dampens the risky investment because of the reduction in free wealth and makes the portfolio lean towards the bond because the stock can be used neither for hedging purpose nor ensuring future minimum consumption. Panel (c) illustrates the horizon effect. It turns out that while the stock demand decreases with investment horizon, the bond demand increases, since longer horizon generates larger value of the habit bond and higher volatility of future investment opportunities. A comparison between Table 3.2 and Table 3.4 shows that for given parameter values, the portfolio share in the stock is higher in the case of only one nominal bond than it is in the complete market. The higher demand for the stock stems from the fact that $\mathrm{d}z_S$ is calibrated to be negatively correlated with both $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$. As the bonds have negative loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$, the correlation between the nominal returns on the stock and the bonds is positive, which dampens the stock investment in the myopic portfolio. The approximately optimal portfolio in this case takes lower interest risk exposure but higher inflation risk exposure than does the optimal portfolio in the complete market. Because the correlation between $\mathrm{d}z_S$ and $\mathrm{d}z_r$ is much higher than that between $\mathrm{d}z_S$ and $\mathrm{d}z_\pi$, the decreased correlation effect associated with lower interest rate exposure outweighs the increased correlation effect associated with higher inflation exposure.

Now we turn to the optimal portfolio strategy under nominal habit formation. Table 3.5 shows the results for different habit parameters and investment horizon. Some interesting changes emerge as compared to the optimal portfolio strategy under real habit formation shown in Table 3.2. First, the impact of inflation on the optimal portfolio is magnified. As shown in panel (a), the presence of habit persistence induces larger inflation risk exposure and this effect intensifies with habit strength, which is in sharp contrast to the decreasing inflation risk exposure in the real habit case. For any given habit strength, the optimal portfolio under nominal habit formation has much larger loadings on the inflation risk than under real habit formation. These distinctions are consequences of different risk profiles of the hedge portfolio and the subsistence portfolio under different types of habit formation: while these two portfolios under real habit formation are only subject to the interest rate risk, those under nominal habit formation carry the expected inflation risk through the habit bond $\hat{f}$, because $\hat{f}$ is comprised of nominal zero-coupon bonds rather than inflation-indexed zero-coupon bonds. Second, although the equity exposure and interest risk exposure remains decreasing in habit strength, they get higher as compared to the real habit case due to the stronger leverage effect: inflation erodes the habit bond price, thereby leaving more free wealth. Third, the optimal demand for the inflation-indexed bond becomes dependent on the habit parameters and investment horizon, because the subsistence portfolio is left uninsured against unexpected inflation. As the habit bond price increases, which is associated with stronger habit persistence and longer horizon, the optimal inflation-index bond holding declines. Fourth, panel (c) shows that the horizon effect on the inflation risk sensitivity is reversed. This is also due to the bigger impact of the inflation risk on the optimal portfolio strategy.

Table 3.6 shows the approximately optimal portfolio strategy with one nominal bond and one inflation-indexed bond under nominal habit formation. Like the results in the real habit formation case, there is only a marginal difference in the risk exposures of the portfolio between complete market and incomplete market settings. The demand for the nominal bond is much larger under nominal habit formation than under real habit formation because as explained above the inflation exposure is increased under nominal habit formation.

Table 3.7 reports the approximately optimal portfolio strategy with one nominal bond under nominal habit formation. A comparison between Table 3.4 and Table 3.7 reveals that the demand for both risky assets is larger under nominal habit formation because of the increase in free wealth. The lower stock-to-bond ratio under nominal habit formation implies that the composition of the portfolio leans more towards the bond in the former setting. This tilt stems from the increases in the hedge demand and subsistence demand induced by the inflation risk.

In order to study how the exclusion of assets influences the welfare of the investors, we calculate the fraction of the initial wealth an investor is willing to give up in order to complete market as the utility costs associated with the market incompleteness. As shown in Table 3.8, the welfare losses resulting from excluding one nominal bond from the asset menu are negligibly small. This is consistent with the marginal difference between the exposures of the optimal portfolio in the complete market case and those in the case of two long-term bonds and implies that the approximate strategy in the latter case is a good substitute for optimal strategy in the former case, which always requires a short position in one of the bonds. In contrast, the welfare losses of having access only to one nominal bond are substantial. They increase with the habit strength and investment horizon, but are hardly affected by the initial habit level. Another interesting observation is that welfare costs are higher under real habit formation than under nominal habit formation.

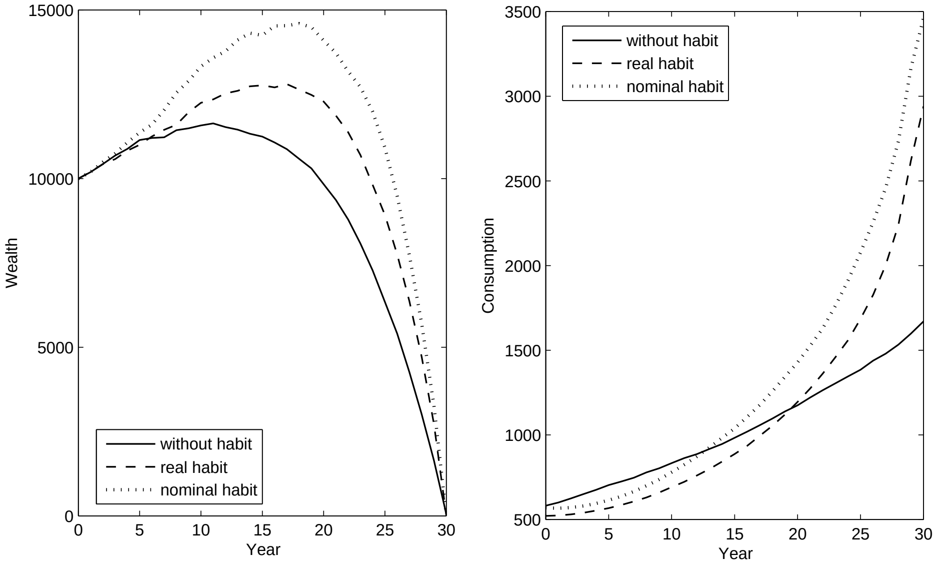
Finally, we investigate the expected wealth and expected consumption under different types of habit formations, which are illustrated in Figure 3.1. Panel (a) shows that all three types of investors accumulate wealth in the early periods and decumulate wealth in the late periods. The accumulation is slowest for the non-habit investor, modest for the real habit investor and fastest for the nominal habit investor. Compared with the habit investors, the non-habit investor does not have to reserve a fraction of wealth for ensuring future subsistence consumption and enjoy higher consumption in the early periods, which is clearly displayed in the right graph. In the late periods, however, the consumption of the habit investors exceeds that of the non-habit investor because of the higher saving rate generated by habit formation. The nominal habit investor has higher wealth and consumption than the real habit investor over the whole life-cycle, both because the nominal habit investor has more free wealth to invest in stocks and benefit more from equity risk premium and because she has a higher marginal propensity to consume on average than the real habit investor.

The wealth accumulation phase arises from the equity risk premium implied by the estimates in Brennan and Xia (2002), which seems unrealistically high in the current market circumstances. Therefore, it is of interest to study the case with lower equity risk premium, which is shown in panel (b). When equity risk premium is set at a lower level, the wealth of three types of investors decumulates over the whole life-cycle. Interestingly, in face of worse market conditions, the habit investors begin with higher consumption than their counterpart, but reduce spending for some periods, because they have to drive down the habit level and increase saving to ensure that future habit consumption can be sustained.

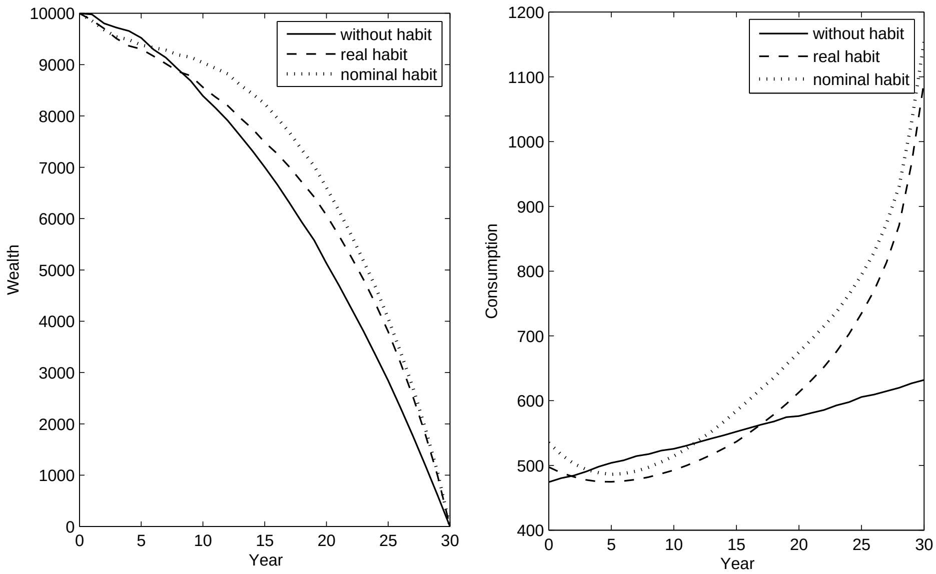
3.5 Conclusion

In this chapter, we have derived the optimal portfolio and consumption policies for an investor with habit formation in preferences and subject to inflation risk. Specifically, we considered two types of habit formation: one is based on past real consumption, while the other on past nominal consumption, which mimic the demand for real guarantees and for nominal guarantees. We also studied the case in which there is only one nominal bond available. The optimal strategy was expressed explicitly in terms of the solution to a linear partial differential equation.

The optimal portfolio is a combination of three portfolios: a myopic portfolio, a hedge portfolio and a subsistence portfolio. The effects of inflation on the optimal



(a) $\lambda_S = 0.343$



(b) $\lambda_S = 0.200$

Figure 3.1: Expected wealth and consumption under different types of habit formation. Panel (a) and (b) show the expected wealth and expected consumption for high equity risk premium ($\lambda_S = 0.358$) and low equity risk premia ($\lambda_S = 0.200$), respectively. In each panel, the left graph plots the expected wealth and the right graph plots the expected consumption. The solid line is for the case without habit formation, the dashed line is for the case with real habit formation and the dotted line is for the case with nominal habit formation.

strategy turn out to depend on the type of habit formation. Under real habit formation, the importance of inflation is limited because inflation risk does not affect the formation of real habit level. On the contrary, inflation risk plays a much bigger role in the case of nominal habit formation, because it alters the risk characteristics of both the hedge demand and subsistence demand and, consequently, the inflation risk exposure of the overall portfolio is raised. Moreover, while the optimal portfolio takes a full hedge against unexpected inflation risk under real habit formation, it leaves the subsistence portfolio uninsured under nominal habit formation. The dependence on the type of habit formation is robust to the incompleteness of the financial market. Another interesting observation in the case of one bond is that the portfolio is tilted more towards bonds under nominal habit formation than under real habit formation.

There are several avenues for future research. First, the dependence of optimal strategy on the type of habit formation raises a need to empirically test whether and to what extent households have money illusion in forming their habit. Although there are a bunch of papers providing evidence in support of habit formation, to the best of our knowledge none of them takes into account households' attitude towards inflation. Second, as human capital is a large component of households' wealth, it is interesting to add labor income to this model and study the optimal strategy for habit-households in the wealth accumulation phase, which contrasts with the wealth decumulation setting assumed in this chapter.

3.6 Appendix

3.6.1 Proof of Theorem 1

We first derive the solution to the dual problem without habit formation formulated in Schroder and Skiadas (2002),

$$\max_{C_t} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{(\frac{\hat{C}_t}{\Pi_t})^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{3.54}$$

$$\text{s.t.} \quad \mathrm{E}\left[\int_0^T \frac{\hat{m}_t}{\hat{m}_0}\frac{\hat{C}_t}{\Pi_t}\mathrm{d}t\right] \leq \frac{\hat{W}_0}{\Pi_0}. \tag{3.55}$$

The dual problem is closely related to the one solved by Brennan and Xia (2002).[13] The only difference is the introduction of an inflation-index bond, which serves to complete the market. Following Brennan and Xia (2002), one can solve the dual problem using the martingale approach. The indirect utility function is,

$$\hat{J}_t = \frac{\hat{Q}_t^{\gamma}\hat{w}_t^{1-\gamma}}{1-\gamma}, \tag{3.56}$$

the optimal dual consumption strategy is

$$\hat{c}_t = \frac{\hat{w}_t}{\hat{Q}_t}, \tag{3.57}$$

and the optimal dual portfolio strategy is

$$\hat{x}_t^* = (\sigma')^{-1}\left(\frac{-\hat{\phi}}{\gamma}\right) + (\sigma')^{-1}(\sigma_{\hat{Q}t} + \xi), \tag{3.58}$$

where $\sigma_{\hat{Q}}$ is the percentage volatility vector of the process $\hat{Q}$ defined by

$$\hat{Q}_t = \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\frac{\delta(s-t)}{\gamma}}\left(\frac{\hat{m}_s}{\hat{m}_t}\right)^{1-\frac{1}{\gamma}}\mathrm{d}s\right]. \tag{3.59}$$

It follows from Schroder and Skiadas (2002) that there is an isomorphism between

[13] The variables in dual economy are denoted by hat.

the primal problem with linear habit formation and the dual primal problem without habit formation:

$$\hat{c}_t = c_t - h_t, \tag{3.60}$$

$$\hat{w}_t = \frac{w_t - h_t f_t}{1 + \alpha f_t}, \tag{3.61}$$

$$\hat{m}_t = m_t(1 + \alpha f_t), \tag{3.62}$$

$$\hat{\phi} = \phi + \frac{\alpha f_t}{1 + \alpha f_t}\sigma_{ft}. \tag{3.63}$$

It is important to note that $\hat{m}$, which corresponds to the habit-adjusted pricing kernel, depends on habit formation via f. Substituting (3.62) into (3.59) yields

$$\hat{Q}_t = g_t(1 + \alpha f_t)^{\frac{1-\gamma}{\gamma}}, \tag{3.64}$$

where g has to be solved numerically. The PDE for g in (3.33) follows from Equation (16) in Munk (2008).

Using (3.56), (3.60) and (3.61), the indirect utility function in the primal problem can be obtained

$$\begin{aligned} J_t &= \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\delta(s-t)}\frac{(c_s^* - h_s)^{1-\gamma}}{1-\gamma}\mathrm{d}s\right] \\ &= \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\delta(s-t)}\frac{(\hat{c}_s^*)^{1-\gamma}}{1-\gamma}\mathrm{d}s\right] \\ &= \hat{J}_t \\ &= \frac{g_t^{\gamma}(w_t - h_t f_t)^{1-\gamma}}{1-\gamma} \end{aligned} \tag{3.65}$$

Proposition 1 in Schroder and Skiadas (2002) shows that if $\hat{c}^*$ is the optimal primal consumption strategy, the dual optimal consumption strategy $\hat{c}$ satisfies,

$$c_t^* = h_t + \frac{\hat{c}_t^*}{\hat{w}_t}\frac{w_t - h_t f_t}{1 + \alpha f_t}. \tag{3.66}$$

Combining (3.57), (3.64) and (3.66) yields the optimal primal consumption strategy c^* in (4.39).

Following Proposition 8 in Schroder and Skiadas (2002), one can derive the relationship between the optimal primal and dual portfolio strategy in the presence of inflation under real habit formation. Proposition 1 and Proposition 3 in Schroder and Skiadas (2002) imply

$$m_t\left(w_t+\frac{h_t}{\alpha}\right)=\hat{m}_t\left(\hat{w}_t+\frac{h_t}{\alpha}\right) \tag{3.67}$$

Using integration by parts, we obtain

$$m_t \mathrm{d}w_t+\left(w_t+\frac{h_t}{\alpha}\right)\mathrm{d}m_t=\hat{m}_t \mathrm{d}\hat{w}_t+\left(\hat{w}_t+\frac{h_t}{\alpha}\right)\mathrm{d}\hat{m}_t+\mathrm{d}(BV)_t. \tag{3.68}$$

Dividing both sides of (3.68) and using the fact (from Proposition 3) that $(1+\alpha\hat{f})(\hat{w}/w)=1-(h\hat{f})$, we obtain

$$\frac{\mathrm{d}w_t}{w_t}=\left(1-\frac{h_t\hat{f}_t}{w_t}\right)\frac{\mathrm{d}\hat{w}_t}{\hat{w}_t}+\left(1+\frac{h_t}{\alpha w_t}\right)(\hat{\phi}'-\phi')\mathrm{d}z_t+\mathrm{d}(BV)_t \tag{3.69}$$

On the other hand, the budget equations in the primal and dual markets imply

$$\frac{\mathrm{d}w_t}{w_t}=\mathrm{d}(BV)_t+(x_t'\sigma-\xi')\mathrm{d}z_t \tag{3.70}$$

$$\frac{\mathrm{d}\hat{w}_t}{\hat{w}_t}=\mathrm{d}(BV)_t+(\hat{x}_t'\sigma-\xi')\mathrm{d}z_t. \tag{3.71}$$

Combining the last three equations, and matching the martingale parts yields,

$$x_t^*=\left(1-\frac{h_tf_t}{w_t}\right)\hat{x}_t^*+\left(1+\frac{h_t}{\alpha w_t}\right)(\sigma')^{-1}\frac{\alpha f_t}{1+\alpha f_t}\sigma_{ft}+\frac{h_tf_t}{w_t}(\sigma')^{-1}\xi. \tag{3.72}$$

It is straightforward to verify that

$$\sigma_{\hat{Q}t}=\sigma_{gt}+\left(\frac{1}{\gamma}-1\right)\frac{\alpha f_t}{1+\alpha f_t}\sigma_{ft}. \tag{3.73}$$

Substituting this into $\hat{x}$ and the resulting expression into (3.72), we obtain (3.35).

3.6.2 Proof of Theorem 2

We follow De Jong (2008) in deriving the approximate solution in the incomplete market. As shown in Brennan and Xia (2002), the optimal real wealth in the dual problem evolves as,

$$\mathrm{d}\ln w_t = \mathrm{d}(BV)_t + \left(-\frac{\hat{\phi}'}{\gamma} + \sigma'_{\hat{Q}t}\right)\mathrm{d}z_t \tag{3.74}$$

where BV is short for some bounded variation process and can be different in each occurrence of abbreviation. On the other hand, if $\hat{x}$ is the vector of the portfolio weights to the stock and the bond in the dual problem, the real wealth process is given by

$$\mathrm{d}\ln w_t = \mathrm{d}(BV)_t + (\hat{x}'_t\sigma_I - \xi')\,\mathrm{d}z_t \tag{3.75}$$

Equating (3.74) and (3.75) yields the optimal portfolio strategy in the complete market. However, the optimal wealth process can not be achieved by dynamic trading in the available assets. Instead, we derive an approximately optimal dual portfolio by minimizing the norm of the difference between the optimal and feasible wealth dynamics

$$\min \quad \left|\left| \left(\hat{x}'_t\sigma_I - \xi' + \frac{\hat{\phi}'}{\gamma} - \sigma'_{\hat{Q}t}\right)\mathrm{d}z_t\right|\right| \tag{3.76}$$

with $||a'\mathrm{d}z|| = a'\rho a$. The first order condition is

$$\sigma_I\rho\left(\sigma'_I\hat{x}_t - \xi + \frac{\hat{\phi}}{\gamma} - \sigma_{\hat{Q}t}\right) = 0, \tag{3.77}$$

and therefore the optimal dual portfolio strategy is given by

$$\hat{x}_t = (\sigma_I\rho\sigma'_I)^{-1}\left(-\sigma_I\rho\frac{\hat{\phi}}{\gamma} + \sigma_I\rho(\sigma_{\hat{Q}t} + \xi)\right). \tag{3.78}$$

To obtain the optimal primal portfolio strategy, we minimize the norm of the difference between the optimal strategy in two economies[14],

$$\min \quad || \left(x_t' \sigma_I - \left(1 - \frac{h_t f_t}{w_t}\right) \hat{x}_t' \sigma_I - \left(1 + \frac{h_t}{\alpha w_t}\right) \frac{\alpha f_t}{1 + \alpha f_t} \sigma_{ft}' - \frac{h_t f_t}{w_t} \xi' \right) \mathrm{d}z_t ||. \tag{3.79}$$

The first order condition is

$$\sigma_I \rho \left(\sigma_I' x_t - \left(1 - \frac{h_t f_t}{w_t}\right) \sigma_I' \hat{x}_t - \left(1 + \frac{h_t}{\alpha w_t}\right) \frac{\alpha f_t}{1 + \alpha f_t} \sigma_{ft} - \frac{h_t f_t}{w_t} \xi \right) = 0, \tag{3.80}$$

and therefore the approximately optimal portfolio strategy in the incomplete market is given by

$$x_t = \frac{w_t - h_t f_t}{w_t} \hat{x}_t + \left(1 + \frac{h_t}{\alpha w_t}\right) \frac{\alpha f_t}{1 + \alpha f_t} (\sigma_I \rho \sigma_I')^{-1} \sigma_I \rho \sigma_{ft} + \frac{h_t f_t}{w_t} (\sigma_I \rho \sigma_I')^{-1} \sigma_I \rho \xi. \tag{3.81}$$

Plugging (4.40) into (3.78) and the resulting expression into (3.81) yields (3.36).

3.6.3 Proof of Theorem 3

The proof is similar to that of Theorem 1, except with the isomorphism under real habit formation replaced by the isomorphism under nominal habit formation[15], which is shown as follows,

$$\hat{c}_t = c_t - h_t, \tag{3.82}$$

$$\hat{w}_t = \frac{w_t - h_t \hat{f}_t}{1 + \alpha \hat{f}_t}, \tag{3.83}$$

$$\hat{m}_t = m_t (1 + \alpha \hat{f}_t), \tag{3.84}$$

$$\hat{\phi} = \phi + \frac{\alpha \hat{f}_t}{1 + \alpha \hat{f}_t} \sigma_{\hat{f}t}. \tag{3.85}$$

It should be noted that the primal problem with real habit formation and that with nominal habit formation are associated with the same dual problem. Based on the

[14]The relationship between the optimal strategy is discussed in Schroder and Skiadas (2002).

[15]The proof of the isomorphism under nominal habit formation in the presence of inflation risk is available upon request.

solution to the dual problem derived in the proof of Theorem 1, we have

$$\hat{Q}_t = \hat{g}_t (1 + \alpha \hat{f}_t)^{\frac{1-\gamma}{\gamma}}. \tag{3.86}$$

Again, the PDE for $\hat{g}$ in (3.46) follows from Equation (16) in Munk and Sørensen (2004). It is straightforward to verify that the indirect utility function and the optimal consumption strategy in the primal problem are given by

$$J_t = \frac{\hat{g}_t^{\gamma} (w_t - h_t \hat{f}_t)^{1-\gamma}}{1-\gamma}, \tag{3.87}$$

$$c_t^* = h_t + (1 + \alpha \hat{f}_t)^{-\frac{1}{\gamma}} \frac{w_t^* - h_t \hat{f}_t}{\hat{g}_t}. \tag{3.88}$$

Similar to the real habit case, one can show under nominal habit formation, the relationship between the optimal dual portfolio strategy $\hat{x}^*$ and the optimal primal portfolio strategy x^* is given by

$$x_t^* = \left(1 - \frac{h_t \hat{f}_t}{w_t}\right) \hat{x}_t^* + \left(1 + \frac{h_t}{\alpha w_t}\right) (\sigma')^{-1} \frac{\alpha \hat{f}_t}{1 + \alpha \hat{f}_t} \sigma_{\hat{f}t}. \tag{3.89}$$

It is straightforward to verify that

$$\sigma_{\hat{Q}t} = \sigma_{\hat{g}t} + \left(\frac{1}{\gamma} - 1\right) \frac{\alpha \hat{f}_t}{1 + \alpha \hat{f}_t} \sigma_{\hat{f}t}. \tag{3.90}$$

Substituting this into $\hat{x}$ and the resulting expression into (3.89), we obtain (3.48).

3.6.4 Proof of Theorem 4

The proof is similar to that of Theorem 2, except with the relationship between the strategies in two economies under real habit formation replaced with that under nominal habit formation as shown in (3.89).

Table 3.2: Optimal portfolio strategy in complete market under real habit formation

(a) For different habit strength ($\beta - \alpha$)

$\beta - \alpha$	L_r	L_π	x_S	x_{N1}	x_{N2}	x_I
0.1	-9.027	-1.969	0.484	5.726	-0.420	1.000
0.2	-9.541	-2.348	0.578	6.689	-0.485	1.000
0.3	-9.752	-2.501	0.620	7.093	-0.513	1.000
0.4	-9.868	-2.592	0.638	7.336	-0.531	1.000
No habit	-10.597	-2.752	0.703	9.759	-0.785	1.000

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_{N1}	x_{N2}	x_I
200	-10.121	-2.438	0.593	8.822	-0.718	1.000
300	-9.571	-2.188	0.535	7.274	-0.569	1.000
400	-9.027	-1.969	0.484	5.726	-0.420	1.000
500	-8.478	-1.754	0.431	4.178	-0.271	1.000
600	-7.928	-1.531	0.384	2.630	-0.122	1.000

(c) For different investment horizon T

T	L_r	L_π	x_S	x_{N1}	x_{N2}	x_I
1	-4.542	-2.751	0.678	-14.255	1.919	1.000
5	-5.433	-2.432	0.601	-9.807	1.381	1.000
10	-6.506	-2.203	0.539	-4.887	0.801	1.000
20	-8.061	-2.018	0.501	1.768	0.031	1.000
30	-9.027	-1.969	0.484	5.726	-0.420	1.000

The table shows the optimal portfolio strategy in complete market under real habit formation. L_r (L_π) is the sensitivity of the portfolio to the real interest rate risk (expected inflation risk). x_S , x_{N1}, x_{N2}, x_I are the fractions of wealth invested in the stock, the 1-year nominal bond, the 10-year nominal bond and the 10-year inflation-indexed bond respectively. The parameter values are as follows: $\gamma = 3$, $\delta = 0.02$, $W_0 = 10000$, $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Table 3.3: Approximately optimal portfolio strategy with one 10-year nominal bond and one 10-year inflation-indexed bond under real habit formation

(a) For different habit strength $(\beta - \alpha)$

$\beta - \alpha$	L_r	L_π	x_S	x_N	x_I
0.1	-8.970	-1.989	0.485	0.227	1.222
0.2	-9.468	-2.370	0.578	0.271	1.259
0.3	-9.676	-2.527	0.616	0.288	1.275
0.4	-9.793	-2.611	0.636	0.298	1.284
No habit	-10.537	-2.786	0.703	0.318	1.384

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_N	x_I
200	-10.040	-2.444	0.594	0.279	1.343
300	-9.505	-2.216	0.539	0.253	1.282
400	-8.970	-1.989	0.485	0.227	1.222
500	-8.434	-1.762	0.430	0.201	1.161
600	-7.899	-1.534	0.375	0.175	1.101

(c) For different investment horizon T

T	L_r	L_π	x_S	x_N	x_I
1	-4.625	-2.699	0.675	0.308	0.439
5	-5.487	-2.394	0.596	0.273	0.613
10	-6.531	-2.181	0.540	0.249	0.806
20	-8.034	-2.023	0.496	0.231	1.067
30	-8.970	-1.989	0.485	0.227	1.222

The table shows the approximately optimal portfolio strategy with one 10-year nominal bond and one 10-year inflation-indexed bond under real habit formation. L_r (L_π) is the sensitivity of the portfolio to the real interest rate risk (expected inflation risk). x_S , x_N, x_I are the fractions of wealth invested in the stock, the 10-year nominal bond and the 10-year inflation-indexed bond, respectively. The parameter values are as follows: $\gamma = 3$, $\delta = 0.02$, $W_0 = 10000$, $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Table 3.4: Approximately optimal portfolio strategy with one 10-year nominal bond under real habit formation

(a) For different habit strength ($\beta - \alpha$)

$\beta - \alpha$	L_r	L_π	x_S	x_N	x_S/x_N
0.1	-3.559	-5.038	0.535	0.575	0.931
0.2	-3.894	-5.512	0.630	0.629	1.002
0.3	-4.032	-5.708	0.669	0.651	1.027
0.4	-4.108	-5.815	0.690	0.664	1.040
No habit	-4.411	-6.244	0.759	0.712	1.066

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_N	x_S/x_N
200	-4.094	-5.795	0.650	0.661	0.983
300	-3.826	-5.416	0.593	0.618	0.959
400	-3.559	-5.038	0.535	0.575	0.931
500	-3.292	-4.659	0.478	0.532	0.900
600	-3.024	-4.281	0.421	0.488	0.862

(c) For different investment horizon T

T	L_r	L_π	x_S	x_N	x_S/x_N
1	-2.681	-3.795	0.693	0.433	1.602
5	-2.772	-3.924	0.622	0.448	1.389
10	-2.962	-4.193	0.574	0.478	1.200
20	-3.310	-4.685	0.541	0.535	1.011
30	-3.559	-5.038	0.535	0.575	0.931

The table shows the approximately optimal portfolio strategy with one 10-year nominal bond under real habit formation. L_r (L_π) is the sensitivity of the portfolio to the real interest rate risk (expected inflation risk). x_S (x_N) is the fraction of wealth invested in the stock (the 10-year nominal bond). x_S/x_N is the stock-to-bond ratio. The parameter values are as follows: $\gamma = 3$, $\delta = 0.02$, $W_0 = 10000$, $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Table 3.5: Optimal portfolio strategy in complete market under nominal habit formation

(a) For different habit strength ($\beta - \alpha$)

$\beta - \alpha$	L_r	L_π	x_S	x_{N1}	x_{N2}	x_I
0.1	-9.711	-4.466	0.549	6.802	-0.256	0.782
0.2	-9.911	-3.617	0.602	8.116	-0.501	0.857
0.3	-10.021	-3.257	0.628	8.664	-0.603	0.894
0.4	-10.103	-3.078	0.644	8.946	-0.656	0.916
No habit	-10.597	-2.752	0.703	9.759	-0.785	1.000

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_{N1}	x_{N2}	x_I
200	-10.512	-4.318	0.626	7.710	-0.375	0.891
300	-10.112	-4.391	0.588	7.256	-0.316	0.836
400	-9.701	-4.466	0.549	6.802	-0.256	0.782
500	-9.304	-4.538	0.511	6.347	-0.197	0.727
600	-8.911	-4.614	0.454	5.893	-0.138	0.673

(c) For different investment horizon T

T	L_r	L_π	x_S	x_{N1}	x_{N2}	x_I
1	-4.536	-2.798	0.680	-13.452	1.833	0.963
5	-5.513	-3.039	0.614	-7.958	1.243	0.867
10	-6.701	-3.548	0.568	-3.380	0.786	0.814
20	-8.514	-4.218	0.552	2.634	0.185	0.786
30	-9.711	-4.466	0.549	6.802	-0.256	0.782

The table shows the optimal portfolio strategy in complete market under nominal habit formation. L_r (L_π) is the sensitivity of the portfolio to the real interest rate risk (expected inflation risk). x_S , x_{N1}, x_{N2}, x_I are the fractions of wealth invested in the stock, the 1-year nominal bond, the 10-year nominal bond and the 10-year inflation-indexed bond respectively. The parameter values are as follows: $\gamma = 3$, $\delta = 0.02$, $W_0 = 10000$, $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Table 3.6: Approximately optimal portfolio strategy with one 10-year nominal bond and one 10-year inflation-indexed bond under nominal habit formation

(a) For different habit strength ($\beta - \alpha$)

$\beta - \alpha$	L_r	L_π	x_S	x_N	x_I
0.1	-9.669	-4.489	0.550	0.512	1.050
0.2	-9.861	-3.646	0.603	0.416	1.177
0.3	-9.974	-3.294	0.629	0.376	1.235
0.4	-10.051	-3.113	0.644	0.355	1.268
No habit	-10.537	-2.786	0.703	0.318	1.384

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_N	x_I
200	-10.447	-4.349	0.626	0.496	1.194
300	-10.068	-4.419	0.588	0.504	1.122
400	-9.669	-4.489	0.550	0.512	1.050
500	-9.271	-4.560	0.511	0.520	0.977
600	-9.872	-4.630	0.473	0.528	0.905

(c) For different investment horizon T

T	L_r	L_π	x_S	x_N	x_I
1	-4.628	-2.748	0.676	0.314	0.434
5	-5.556	-3.010	0.609	0.343	0.554
10	-6.720	-3.542	0.572	0.404	0.681
20	-8.487	-4.231	0.552	0.483	0.890
30	-9.669	-4.489	0.550	0.512	1.050

The table shows the approximately optimal portfolio strategy with one 10-year nominal bond and one 10-year inflation-indexed bond under nominal habit formation. L_r (L_π) is the sensitivity of the portfolio to the real interest rate risk (expected inflation risk). x_S , x_N, x_I are the fractions of wealth invested in the stock, the 10-year nominal bond and the 10-year inflation-indexed bond respectively. The parameter values are as follows: $\gamma = 3$, $\delta = 0.02$, $W_0 = 10000$, $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Table 3.7: Approximately optimal portfolio strategy with one 10-year nominal bond under nominal habit formation

(a) For different habit strength ($\beta - \alpha$)

$\beta - \alpha$	L_r	L_π	x_S	x_N	x_S/x_N
0.1	-5.022	-7.108	0.593	0.811	0.732
0.2	-4.650	-6.582	0.652	0.751	0.868
0.3	-4.505	-6.378	0.680	0.728	0.935
0.4	-4.435	-6.278	0.697	0.716	0.973
No habit	-4.411	-6.244	0.759	0.712	1.066

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_N	x_S/x_N
200	-5.178	-7.329	0.676	0.836	0.809
300	-5.100	-7.219	0.635	0.824	0.771
400	-5.022	-7.108	0.593	0.811	0.732
500	-4.944	-6.998	0.552	0.798	0.691
600	-4.866	-6.887	0.511	0.786	0.650

(c) For different investment horizon T

T	L_r	L_π	x_S	x_N	x_S/x_N
1	-2.706	-3.831	0.694	0.437	1.588
5	-3.103	-4.392	0.632	0.501	1.261
10	-3.703	-5.242	0.600	0.598	1.003
20	-4.557	-6.451	0.596	0.736	0.801
30	-5.022	-7.108	0.593	0.811	0.732

The table shows the approximately optimal portfolio strategy with one 10-year nominal bond under nominal habit formation. L_r (L_π) is the sensitivity of the portfolio to the real interest rate risk (expected inflation risk). x_S (x_N) is the fraction of wealth invested in the stock (the 10-year nominal bond). x_S/x_N is the stock-to-bond ratio. The parameter values are as follows: $\gamma = 3$, $\delta = 0.02$, $W_0 = 10000$, $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Table 3.8: Welfare costs of market incompleteness

	(a) For different habit strength ($\beta - \alpha$)				
	0.1	0.2	0.3	0.4	No habit
Real habit formation					
10-year nominal and 10-year indexed	0.05%	0.02%	0.00%	0.00%	0.00%
10-year nominal only	6.72%	4.76%	3.66%	2.66%	0.99%
Nominal habit formation					
10-year nominal and 10-year indexed	0.02%	0.01%	0.00%	0.00%	0.00%
10-year nominal only	2.53%	2.44%	1.86%	1.19%	0.99%
	(b) For different initial habit level h_0				
	200	300	400	500	600
Real habit formation					
10-year nominal and 10-year indexed	0.05%	0.05%	0.05%	0.05%	0.05%
10-year nominal only	6.67%	6.70%	6.72%	6.78%	6.85%
Nominal habit formation					
10-year nominal and 10-year indexed	0.02%	0.02%	0.02%	0.02%	0.02%
10-year nominal only	2.48%	2.50%	2.53%	2.55%	2.58%
	(c) For different investment horizon T				
	1 year	5 years	10 years	20 years	30 years
Real habit formation					
10-year nominal and 10-year indexed	0.00%	0.01%	0.03%	0.04%	0.05%
10-year nominal only	0.00%	0.04%	1.96%	4.90%	6.72%
Nominal habit formation					
10-year nominal and 10-year indexed	0.00%	0.00%	0.00%	0.00%	0.02%
10-year nominal only	0.00%	0.09%	1.07%	1.41%	2.53%

The table shows the welfare costs of market incompleteness. Welfare costs are determined as the fraction of the initial wealth an investor is willing to give up in order to have a complete market. The parameter values are as follows: $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 3.1.

Chapter 4

Portfolio Choice over the Life-Cycle in the Presence of Cointegration between Labor Income and Inflation[1]

We study portfolio choice for a finite-horizon investor whose labor income is cointegrated with inflation. We show that this long-run relationship has substantial impacts on the riskiness of human capital and consequently on the optimal portfolio strategy. Because cointegration raises the long-run correlation between human capital and inflation, young investors' human capital effectively hedges inflation risk and crowds out the allocation to inflation-indexed bonds. However, the hedging power of human capital diminishes for older investors because of a weaker cointegration effect and less importance of human capital in total wealth. These effects together show that inflation-indexed bonds matter more for older investors than for young investors.

4.1 Introduction

Hedging inflation risk is a concern of critical importance for households, because typically they have long investment horizons and consider inflation as a direct threat to the purchasing power of their wealth. Extensive studies have already been been carried out to assess the inflation hedging power of a variety of assets, such as stocks, nominal

[1]This chapter is based on Zhou (2014)

bonds, treasury inflation protected securities (TIPS), real estates, and commodities.[2] Nonetheless, human capital, despite its great importance, has received little attention so far in this regard.[3] For many households, especially for young households, human capital is their largest asset. Its risk characteristics and, in particular, its inflation hedging ability is bounded to have large impact on the optimal portfolio strategy of the households. Moreover, the declining importance of human capital in total wealth over the life-cycle may remarkably differentiate the optimal portfolio strategy for the young and that for the old. Since this key life-cycle implication for inflation hedging is unlikely to be derived using traditional framework without labor income, this chapter solve a life-cycle model of optimal portfolio and consumption choice with inflation risk and stochastic labor income.

A salient and important feature of our model is that we allow for a cointegration relationship between labor income and inflation. Specifically, following Benzoni, Collin-Dufresne, and Goldstein (2007), we split the investor's labor income into two components: the one is aggregate labor income component, which is specified to be cointegrated with realized inflation, and the other is idiosyncratic labor income component, which captures the hump-shaped pattern of individual labor income over the life-cycle and idiosyncratic labor income shocks. For the contemporaneous correlations between nominal labor income and inflation, we consider two cases: the one assumes a positive correlation and the other assumes that nominal labor income does not respond to inflation shocks. We refer to the former case as partial wage rigidity and the latter case as nominal wage rigidity. In the absence of idiosyncratic labor income risks and portfolio constraints, we derive an explicit solution for the optimal portfolio and consumption decisions using martingale approach. Based on this solution, we explore the influence of the cointegration relationship on the inflation hedging property of human capital and consequently on the optimal portfolio strategy for households.

The relationship between labor income and inflation has long been studied in macroeconomic literature. Many studies investigate the responsiveness of wages to variations in macroeconomic conditions and document ample evidence in support of nominal wage

[2] See, for example, Bodie (1976), Fama and Schwert (1977), Kaul (1987), Hartzell, Hekman, and Miles (1987), Boudoukh and Richardson (1993), Schotman and Schweitzer (2000), Anari and Kolari (2002), Amenc, Martellini, and Ziemann (2009), Bekaert and Wang (2010) and Boons, De Roon, and Szymanowska (2012).

[3] To the best of our knowledge, Fama and Schwert (1977) is the only paper that investigates the effectiveness of labor income as a inflation hedge.

rigidity; See, for instance, Bernanke and Carey (1996), Kahn (1997), Smith (2000) and Nickell and Quintini (2003). These results imply that nominal labor income has a low contemporaneous correlation with inflation and provides a poor inflation hedge, which is consistent with the finding of Fama and Schwert (1977) that labor income exhibits a low short-term correlation with either expected or unexpected inflation. On the other hand, motivated by the expectations-augmented Phillips-curve model, which contends the labor incomes and prices are mutually causal, researchers have focused a great deal of attention on the long-run relationship between labor income and inflation and found evidence in favor of cointegration; See, for instance, Mehra (1991), Ghali (1999), Banerjee and Russell (2001) and Banerjee, Cockerell, and Russell (2001). This finding has important implications for inflation hedging: even if the contemporaneous correlation between labor income and inflation is low, the correlation between human capital and inflation can be significantly higher due to the long-run cointegration, thereby leading to stronger inflation protection.

Using postwar data from 1947-2012, we find evidence that aggregate labor income and inflation are cointegrated. We acknowledge that we cannot reject the unit root hypothesis for the whole sample period from 1929-2012. However, econometrically it is difficult to distinguish between these two hypotheses as the ADF test is notoriously lacking in power. Moreover, as documented in macroeconomic literature, there might be a structural break in the behavior of labor income around World War II; See, for example, Bernanke and Powell (1986), Hanes (1996) and Basu and Taylor (1999). It is still worth investigating the implications of such a cointegration relationship for the optimal portfolio strategy for two reasons. First, macroeconomic theory and empirical evidence indicate that cointegration is economically plausible. Second, as shown below, the models without cointegration ($\kappa_y = 0$) and with weak cointegration ($\kappa_y = 0.05$) generate notably different inflation hedging strategy, especially for young investors. Since such a low mean reversion is difficult to be detected, rejecting the model with cointegration seems arbitrary. Therefore, we consider both models and compare the qualitative and quantitative properties of the optimal strategy in different models.

The main results are as follows. The cointegration between labor income and inflation is crucial for the risk characteristics of human capital, because it substantially increases the long-run correlation between human capital and realized inflation and, consequently, strengthens the inflation hedging power of human capital. The intuition is that the negative impact of instantaneous inflation shocks on the value of human capital is mitigated by the rebound in the expected growth rate of labor income caused by the long-run dependence. This compensation effect is more pronounced for stronger cointegration relationship. Moreover, the uncertainty in the discount rate produces enormous interest rate risk bearing in human capital. This exposure, however, is independent of the cointegration relationship.

In the explicit solution, the optimal portfolio is composed of three components: (1) a nominal mean-variance tangency portfolio, (2) a hedge portfolio against variation of future investment opportunities and, (3) a correction portfolio for the implicit investment through nominal human capital. Comparison with Brennan and Xia (2002) reveals that the introduction of labor income creates a new correction portfolio that depends on the riskiness of human capital and the relative importance of human capital in total wealth. Because of cointegration, young investors' human capital effectively hedges inflation risk and substitutes for the long position in the inflation-indexed bond. However, this crowding-out effect vanishes for older investors because the ratio of human capital to total wealth declines and the cointegration effect wears off. Consequently, the demand for the inflation-indexed bond is increasing over the life-cycle, which demonstrates that the inflation-indexed bond matters more for older investors than for young investors. The absolute real interest rate exposure exhibits a decreasing pattern over the life-cycle because of the reduction in the implicit interest rate exposure provided by human capital. This contrasts with Brennan and Xia (2002), who find that the absolute real interest rate exposure increases with the investment horizon and this horizon effect disappears quickly. Moreover, the horizon effect on the absolute expected inflation exposure is negative, which is markedly different from their results of no horizon effect. Finally, while the cointegration relationship brings major changes to the expected and unexpected risk taking, it hardly affects the equity and interest risk exposures.

In addition to the literature on the effectiveness of various financial assets in hedging inflation risk sketched above, this article also relates to the strand of papers on dynamic asset allocation with inflation risk; See, for example, Campbell and Viceira (2001), Brennan and Xia (2002), Munk and Sørensen (2004), Koijen, Nijman, and Werker (2010) and Van Hemert (2010). The closest to this chapter are Campbell and Viceira (2001) and Brennan and Xia (2002). Campbell and Viceira (2001) develop an approximately optimal portfolio strategy for an infinite-horizon investor in the presence of interest rate risk and inflation risk and find that long-term inflation-indexed bonds are the most suitable assets for conservative inventors. Brennan and Xia (2002) analyze a finite-horizon investor's asset allocation problem under inflation in the absence of inflation-indexed bonds and show that a nominal bond portfolio that has the highest correlation with inflation realization provides the best inflation hedge. Since they abstract from labor income, their policy implications are universal for people of all ages, who differ in the wealth structure, however. In contrast, we focus on the crowding-out effect of human capital on the optimal portfolio strategy and show that the importance of inflation-indexed bonds crucially depends on the age of investors. Moreover, we find that the introduction of labor income leads to much stronger horizon effects than the results of Brennan and Xia (2002). So far, there are only few papers involving both labor income and inflation risk, such as Koijen, Nijman, and Werker (2010) and Van Hemert (2010). However, they simply assume zero correlation between real labor income and realized inflation and implicitly make the labor income streams like inflation-indexed bonds. This assumption contradicts the evidence of nominal wage rigidity documented in the literature and leaves the inflation hedging unimportant.

On the other hand, this chapter also builds on the literature investigating the relation between labor income and investment opportunities. Benzoni, Collin-Dufresne, and Goldstein (2007) impose a long-run cointegration between labor income and stock market and find that such a relation leads to a large reduction in the optimal stock holdings for sufficiently risk-averse investors. Lynch and Tan (2011) utilize dividend yields to capture the cyclicality of both stock market and labor market and show that the dependence of the conditional joint distribution of labor income on the business cycle induces a negative hedge demand for stocks and generates equity holdings that better match those of U.S. households. In contrast, Munk and Sørensen (2010) link labor income to bond market by allowing the expected labor income growth rate to depend on the stochastic short rate and explore how the sign and magnitude of this dependence affect the riskiness of human capital and the optimal portfolio strategy. This paper complements this strand of literature by linking labor income to bond market through inflation and analyzing the inflation hedging property of human capital.

The rest of the chapter is organized as follows. Section 4.2 describes the model and performs the cointegration test. Section 4.3 explains the details of the model calibration. Section 4.4 derives an analytical solution to the portfolio and consumption optimization problem under the simplifying assumption of no unspanned labor income risks and no portfolio constraints. Section 4.5 concludes.

4.2 The Model

4.2.1 Asset Price Dynamics

We follow Brennan and Xia (2002) in modeling the asset price dynamics. There are four variables determining asset prices: the nominal stock price S, the instantaneous real interest rate r, the instantaneous expected inflation π and the commodity price level Π. The term structure is characterized by the real interest rate and expected inflation. For simplicity, we assume that the risk premia on sources of uncertainty are constant. The stock price follows a geometric Brownian motion as in the Black and Scholes (1973) model. The real interest rate and expected inflation follow Ornstein-Uhlenbeck processes as in the Vasicek (1977) model. The realized inflation equals the expected inflation plus an i.i.d. unexpected inflation shock. The equations driving the state variables are given by

$$\frac{\mathrm{d}S_t}{S_t} = (R_t + \sigma_S \lambda_S)\mathrm{d}t + \sigma_S \mathrm{d}z_{St}, \tag{4.1}$$

$$\mathrm{d}r_t = \kappa_r(\bar{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}z_{rt}, \tag{4.2}$$

$$\mathrm{d}\pi_t = \kappa_\pi(\bar{\pi} - \pi_t)\mathrm{d}t + \sigma_\pi \mathrm{d}z_{\pi t}, \tag{4.3}$$

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \mathrm{d}t + \sigma_\Pi \mathrm{d}z_{\Pi t}, \tag{4.4}$$

where R is the nominal interest rate, λ_S the nominal price of equity risk, κ_π and κ_r mean reversion parameters, and $\bar{r}$ and $\bar{\pi}$ unconditional means. σ_S, σ_r, σ_π and σ_Π are the volatilities of the stock return, real interest rate, expected inflation and realized inflation, respectively. z_S, z_r, z_π and z_Π are the standard Brownian motions that drive the stock return, real interest rate, expected inflation and realized inflation, respectively. It is imporatnt to note that throughout this chapter, we use uppercase letters for nominal variables and the corresponding lowercase letters for their real counterparts.

We can orthogonalize equation (4.4) for unexpected inflation:

$$\begin{aligned}\frac{\mathrm{d}\Pi_t}{\Pi_t} &= \pi_t \mathrm{d}t + \xi_S \mathrm{d}z_{St} + \xi_r \mathrm{d}z_{rt} + \xi_\pi \mathrm{d}z_{\pi t} + \xi_u \mathrm{d}z_{ut} \\ &= \pi_t \mathrm{d}t + \xi' \mathrm{d}z_t, \end{aligned} \tag{4.5}$$

where $\mathrm{d}z = (\mathrm{d}z_S, \mathrm{d}z_r, \mathrm{d}z_\pi, \mathrm{d}z_u)'$ denotes the vector of innovations in standard Brownian motions with $\mathrm{d}z_u$[4] orthogonal to $\mathrm{d}z_S$, $\mathrm{d}z_r$, and $\mathrm{d}z_\pi$. The correlation matrix of $\mathrm{d}z$ therefore is

$$\rho = \begin{pmatrix} \{\rho_{S,r,\pi}\}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix}. \tag{4.6}$$

The real pricing kernel of the financial market, m_t, follows a diffusion process:

$$\begin{aligned} \frac{\mathrm{d}m_t}{m_t} &= -r_t \mathrm{d}t + \phi_S \mathrm{d}z_{St} + \phi_r \mathrm{d}z_{rt} + \phi_\pi \mathrm{d}z_{\pi t} + \phi_u \mathrm{d}z_{ut} \\ &= -r_t \mathrm{d}t + \phi' \mathrm{d}z_t, \end{aligned} \tag{4.7}$$

where, $\phi = (\phi_S, \phi_r, \phi_\pi, \phi_u)'$ is the vector of the constant loadings on the stochastic innovations in the economy and determines the market prices of risk, λ_S, λ_r, λ_π and λ_u, which are associated with innovations $\mathrm{d}z_S$, $\mathrm{d}z_r$, $\mathrm{d}z_\pi$ and $\mathrm{d}z_u$, respectively. Brennan and Xia (2002) show that the vector of nominal market prices of risks $\lambda = (\lambda_S, \lambda_r, \lambda_\pi, \lambda_u)'$ and the nominal short-term risk-free rate R are given by

$$\lambda = \rho(\xi - \phi), \tag{4.8}$$

$$R_t = r_t + \pi_t - \xi' \lambda. \tag{4.9}$$

Note that m only governs the financial market, but not the whole economy due to the idiosyncratic risks and frictions in the labor income. Therefore, m cannot be used for valuing labor income streams except for a few cases that will be discussed in detail below.

Brennan and Xia (2002) show that the time t nominal price of a nominal zero-coupon bond maturing at T, denoted by P_t^T, evolves as,

$$\frac{\mathrm{d}P_t^T}{P_t^T} = [R_t - B_r(T-t)\sigma_r\lambda_r - B_\pi(T-t)\sigma_\pi\lambda_\pi]\mathrm{d}t - B_r(T-t)\sigma_r \mathrm{d}z_{rt} - B_\pi(T-t)\sigma_\pi \mathrm{d}z_{\pi t}, \tag{4.10}$$

where

$$B_r(\tau) = \kappa_r^{-1}(1 - e^{-\kappa_r \tau}), \tag{4.11}$$

$$B_\pi(\tau) = \kappa_\pi^{-1}(1 - e^{-\kappa_\pi \tau}), \tag{4.12}$$

[4]Note that in Brennan and Xia (2002), the subscript "u" means unhedgeable. However, as explained below, we consider a complete market in the benchmark model and thus there is no unhedgeable component of inflation risk. We follow this notation for the purpose of comparison.

It is important to note that the return processes of nominal bonds with different maturities only differ in their loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$. Hence, any desired combination of loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$ can be achieved by positions in any two bonds with different maturities. In contrast, the time t real price of an inflation-indexed bond maturing at time T evolves as

$$\frac{\mathrm{d}p_t^T}{p_t^T} = [r_t - B_r(T-t)\sigma_r\bar{\lambda}_r]\mathrm{d}t - B_r(T-t)\sigma_r\mathrm{d}z_{rt}, \tag{4.13}$$

where $\bar{\lambda}_r = -\phi'\rho e_2$ and $e_2 = (0,1,0,0)'$. Applying Itô's Lemma to its nominal value, $P_t^{T*} = \Pi_t p_t^T$, yields its nominal return,

$$\frac{\mathrm{d}P_t^{T*}}{P_t^{T*}} = [r_t + \pi_t - B_r(T-t)\sigma_r\lambda_r]\mathrm{d}t - B_r(T-t)\sigma_r\mathrm{d}z_{rt} + \xi'\mathrm{d}z_t. \tag{4.14}$$

4.2.2 Labor Income Dynamics

Now, we specify the dynamics of nominal labor income process. Following Benzoni, Collin-Dufresne, and Goldstein (2007), we assume the investor's nominal labor income is a product of two components: L_1, the aggregate nominal labor income associated with the agent's career choice, and L_2, the idiosyncratic labor income. Therefore, her log nominal labor income is given by,

$$\log L_t = \log L_{1t} + \log L_{2t}. \tag{4.15}$$

We assume that the aggregate nominal labor income and realized inflation are cointegrated and its dynamics are given by

$$\frac{\mathrm{d}L_{1t}}{L_{1t}} = \left(\pi_t - \kappa_y y_t + \frac{(\psi-\xi)'\rho(\psi+\xi)+\psi_y^2}{2}\right)\mathrm{d}t + \psi'\mathrm{d}z_t + \psi_y\mathrm{d}z_{yt}, \tag{4.16}$$

where $\psi = (\psi_S, \psi_r, \psi_\pi, \psi_u)'$ are the risk exposures of the aggregate nominal labor income to the spanned shocks $\mathrm{d}z$ and $\mathrm{d}z_y$ is the aggregate labor income shock independent of $\mathrm{d}z$[5]. y is the cointegration variable defined as the difference between the logs of the

[5]The convexity term $\frac{(\psi-\xi)'\rho(\psi+\xi)+\psi_y^2}{2}$ is introduced to make sure y has a zero mean.

aggregate nominal labor income and realized inflation:

$$y_t = \log L_{1t} - \log \Pi_t - \widehat{L\Pi}. \tag{4.17}$$

The constant $\widehat{L\Pi}$ is the long-run mean of log aggregate real labor income. y can be interpreted as the demeaned process of log aggregate real labor income. It is easy to verify that y is a mean-reverting process,

$$\mathrm{d}y_t = -\kappa_y y_t \mathrm{d}t + (\psi - \xi)' \mathrm{d}z_t + \psi_y \mathrm{d}z_{yt}, \tag{4.18}$$

The coefficient κ_y plays a key role in this process, because it determines the mean-reversion speed of the aggregate real labor income and, more importantly, the stationarity of the process. We will test the existence of the long-run cointegration in the following subsection. Further, we specify the dynamics of the idiosyncratic labor income component,

$$\frac{\mathrm{d}L_{2t}}{L_{2t}} = g_t \mathrm{d}t + \psi_l \mathrm{d}z_{lt}, \tag{4.19}$$

where $\mathrm{d}z_l$ is the idiosyncratic labor income shock independent of $\mathrm{d}z$ and $\mathrm{d}z_y$. The drift g is a function of age that captures the hump-shaped pattern of individual labor income over the life cycle. In accordance with Cocco, Gomes, and Maenhout (2005), Van Hemert (2010) and Koijen, Nijman, and Werker (2010), we model g as a second-order polynomial in age,

$$g_t = a_0 + a_1 t + a_2 t^2. \tag{4.20}$$

From equation (4.15), (4.16) and (4.19), we can derive the dynamics of nominal labor income and real labor income using Itô's lemma,

$$\frac{\mathrm{d}L_t}{L_t} = \left(\pi_t - \kappa_y y_t + g_t + \frac{(\psi - \xi)' \rho (\psi + \xi) + \psi_y^2}{2} \right) \mathrm{d}t + \psi' \mathrm{d}z_t + \psi_y \mathrm{d}z_{yt} + \psi_l \mathrm{d}z_{lt}, \tag{4.21}$$

$$\frac{\mathrm{d}l_t}{l_t} = \left(-\kappa_y y_t + g_t + \frac{(\psi - \xi)' \rho (\psi - \xi) + \psi_y^2}{2} \right) \mathrm{d}t + (\psi - \xi)' \mathrm{d}z_t + \psi_y \mathrm{d}z_{yt} + \psi_l \mathrm{d}z_{lt}. \tag{4.22}$$

In our main analysis, we assume there is no income in retirement, that is $L_t = 0$ for

$t \in [\hat{T}, T]$. The exposures of nominal labor income to expected and unexpected inflation risks, ψ_π and ψ_u determine the inflation hedge offered by current labor income. The larger ψ_π and ψ_u are, the more closely nominal labor income moves with realized inflation, the better inflation hedge it creates. In an extreme case, where $\psi_\pi = 0$ and $\psi_u = 0$, nominal labor income does not respond to the expected and unexpected inflation shocks and provides no inflation protection. We refer to this case as nominal wage rigidity in the following analysis. It is important to recognize that the growth rate of real labor income process is dependent on the cointegration variable y and stochastic. The inflation risk exposures of y imply that inflation shocks affect real labor income not only through contemporaneous correlation but also by changing its growth rate.

4.2.3 Preferences

We consider an investor with a standard constant relative risk aversion (CRRA) utility function. She has a fixed retirement date $\hat{T}$ and a fixed investment horizon T. The objective of the investor is to maximize her lifetime utility, which is given by

$$\mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t} \frac{\left(\frac{C_t}{\Pi_t}\right)^{1-\gamma}}{1-\gamma} \mathrm{d}t\right] \tag{4.23}$$

where C is the nominal consumption rate, δ is the subjective discount rate and γ is the risk aversion coefficient. Note that as our focus is on the working phase before retirement, we ignore the bequest motive for simplicity. This is consistent with Hurd (1989), who reports empirical evidence that bequest motives are nearly zero in many countries.

4.2.4 Cointegration Test

We close this section with a test of the cointegration between aggregate labor income and inflation. The long-run relationship between labor income and inflation has long been a heated subject in macroeconomic literature. Theoretically, the expectations-augmented Phillips-curve model suggests that there should be a long-run relationship between labor income and inflation, because they are mutually causal. Motivated by this theory, a vast literature has investigated this long-run relationship between labor income and inflation and found evidence in support of cointegration, which includes Mehra (1991), Ghali (1999), Banerjee and Russell (2001) and Banerjee, Cockerell, and Russell (2001). In the same spirit, we employ a parsimonious model to test the cointegration.

Following Lettau and Ludvigson (2001), we define the aggregate labor income as the sum of wages and salaries, transfer payments, and employer contributions for employee pension and insurance, net of employee contributions for social insurance and taxes. We use annual data from 1929 to 2012 to form the aggregate labor income series. The data are from the National Income and Product Accounts (NIPA) tables complied by the Bureau of Economic Analysis. To calculate the per capita nominal labor income, we divide the aggregate labor income series by the population measure reported in the NIPA tables. The inflation data are from the Center for Research in Security Prices (CRSP).

We test the long-run cointegration by checking the stationarity of the cointegration variable y. For this purpose, we perform an augmented Dickey-Fuller (ADF) test by estimating a regression model

$$\Delta y_t = \alpha_0 + \alpha_1 t + \alpha_2 y_{t-1} + \sum_{i=1}^{G} \beta_i \Delta y_{t-i} + \epsilon_t,$$

where $\Delta y_t = y_t - y_{t-1}$ and $y = \log L_{1t} - \log \Pi_t - \widehat{L\Pi}$, using ordinary least squares method (OLS).

Table 4.1 reports the estimation results. In the first row, we fix the time trend coefficient at zero, while in the other rows, we allow for the presence of a time trend, which is customary to do in unit root test. We also consider specifications with different lags. The coefficient α_2 is significant at the 5% level in the case of $G = 0$ and $\alpha_1 = 0$ for the postwar sample period (1947-2012), which suggests that the unit root hypothesis can be rejected. However, this result is not robust to the inclusion of time trend and lags. When we use a longer sample period (1929-2012), we still cannot reject the unit root hypothesis.

In sum, we find evidence in support of the cointegration between aggregate labor income and inflation. We acknowledge that these results are not robust to the alternative sample period from 1929 to 2012. However, labor income process is likely to behave differently between the prewar and postwar periods, which has been well documented in

Table 4.1: ADF test results for the cointegration variable y

	1929-2012		1947-2012	
	α_2	ι	α_2	ι
$G = 0,\ \alpha_1 = 0$	-0.0180	-1.449	-0.0335	-2.999**
$G = 0$	-0.0501	-1.256	-0.0768	-1.754
$G = 1$	-0.0937	-2.631	-0.0879	-1.937
$G = 2$	-0.0853	-2.289	-0.1097	-2.310

This table shows the Augmented Dickey-Fuller (ADF) test results for the cointegration variable y using annual U.S. data on aggregate labor income and inflation over the period from 1929 to 2012. The estimation model is

$$\Delta y_t = \alpha_0 + \alpha_1 t + \alpha_2 y_{t-1} + \sum_{i=1}^{G} \beta_i \Delta y_{t-i} + \epsilon_t,$$

where $\Delta y_t = y_t - y_{t-1}$ and $y_t = \log L_{1t} - \log \Pi_t - \widehat{L\Pi}$. The labor income series is defined as the sum of wages and salaries, transfer payments, and employer contributions for employee pension and insurance, net of employee contributions for social insurance and taxes. The data on labor income are from the National Income and Product Accounts (NIPA) complied by the Bureau of Economic Analysis. The data on inflation are from the Center for Research in Security Prices (CRSP). Columns under 1929-2012 are the results for the whole sample period and columns under 1947-2012 for the postwar sample period. *, ** and *** denote significance at the 10%, 5% and 1% levels, respectively.

macroeconomic literature; See, for example, Bernanke and Powell (1986), Hanes (1996) and Basu and Taylor (1999). Moreover, our estimated values of κ_y are fairly small and imply half lives of many years. Econometrically, it is difficult to distinguish between $\kappa_y = 0$ and, say, $\kappa_y = 0.05$ given only a few decades of data. Therefore, it is not surprising that the cointegration effect is not detected by the ADF test, which is notoriously lacking in power (i.e., it very often tells us there is a unit root when there is no unit root). More importantly, as outlined above, macroeconomic literature lends strong support to the notion that aggregate labor income and inflation are cointegration. Thus, we continue our analysis with the assumption that the cointegration variable y is stationary.

4.3 Model calibration

In this section we report the calibrated parameter values and briefly explain the data and calibration procedure.

1. **Stocks**. We calibrate the parameters governing stock price dynamics to quarterly U.S. data from the CRSP for the sample period from the first quarter of 1959 to the fourth quarter of 2012. The data are the value-weighted returns, including dividends, on an index comprising all NYSE, AMEX and NASDAQ firms. The data range is motivated by the availability of bond yield data.

2. **Term Structure of Interest Rates**. The parameters governing the term structure are estimated using quarterly U.S. data on nominal interest rates and inflation for the same sample period. The inflation data are from CRSP and the nominal bond yields data are from the Federal Reserve Bank of St. Louis. We use six yields in estimation with 3-month, 6-month, 1-year, 3-year, 5-year, 10-year maturities, respectively. We back out the unobserved real interest rate and expected inflation from the data with a Kalman filter technique, and estimate the model using maximum likelihood method.[6] The estimated mean reversion parameters of real interest and expected inflation rate are, $\kappa_r = 0.4148$ and $\kappa_\pi = 0.0544$, which imply half-lives of 1.7 and 12.7 years, respectively. This is consistent with the findings of Campbell and Viceira (2001), Brennan and Xia (2002) and De Jong, Driessen, and

[6]Details on the estimation procedure can be found in Appendix 4.6.1.

Van Hemert (2008). Since the unexpected inflation risk premium cannot be identified only using data on the nominal bonds and the data on inflation-indexed bond are insufficient to estimate it accurately[7], we impose that the reward for bearing unexpected inflation risk is zero ($\phi_u = 0$). This assumption is consistent with the recent literature; see, for instance, Campbell and Viceira (2001), Koijen, Nijman, and Werker (2010) and Van Hemert (2010). The market price of risk parameters, λ_r and λ_π are estimated by matching the average yields of bonds with maturity of 1 and 10 years. This is performed by using formulas derived by Brennan and Xia (2002).

3. **Labor Income Dynamics**. The key parameter for labor income dynamics is κ_y, as it determines the mean reversion speed of the aggregate real labor income. In subsection 4.2.4, we find some evidence in support of the stationarity of y. However, there is enormous variation in the discrete time point estimates of κ_y, ranging from 0.0106 to over 0.1040, due to the inaccurate measurement. Transforming from discrete time to continuous time yields a range from 0.0107 to 0.1098. In what follows κ_y is set equal to 0.05 as the benchmark value.

   To focus on the inflation hedging ability of labor income, we assume $\psi_S = \xi_S$ and $\psi_r = \xi_r$, which implies that real labor income is not correlated with stocks and interest rate process[8]. The imprecise measurement of κ_y prevents us from obtaining an accurate estimate of ψ_π and ψ_u. In our benchmark case, we fix $\psi_\pi = 0.005$ and $\psi_u = 0.01$. The magnitudes of these two parameters are chosen to make sure that real labor income is partially affected by inflation shocks ($\psi_\pi < \xi_\pi$ and $\psi_u < \xi_u$) and the variance of the aggregate real labor income primarily stems from the aggregate labor income shock z_y. We refer to this partial absorption of inflation shocks as partial wage rigidity. In the following analysis, we also consider the extreme case of nominal wage rigidity ($\psi_\pi = 0$ and $\psi_u = 0$).

   The parameters ψ_y and ψ_l are the standard deviations of the idiosyncratic labor income and aggregate labor income, respectively. Following Benzoni, Collin-Dufresne, and Goldstein (2007), we set $\psi_y = 0.05$ and $\psi_l = 0.15$. This calibration

[7] Only as of 1997, treasury inflation-protected securities (TIPS) have been introduced in the United Stated.

[8] This assumption is consistent with Van Hemert (2010) and Koijen, Nijman, and Werker (2010). There are other studies on the correlation between labor income and stocks returns, e.g. Benzoni, Collin-Dufresne, and Goldstein (2007) and Lynch and Tan (2011), and on the correlation between labor income and stochastic interest rates, e.g. Munk and Sørensen (2010).

is consistent with previous studies that model the labor income process of individual households using data from PSID; see, for example, Carroll and Samwick (1997), Gourinchas and Parker (2002) and Cocco, Gomes, and Maenhout (2005). Moreover, we adapt the deterministic labor income profile for high school group of Cocco, Gomes, and Maenhout (2005) to a continuous-time setting. Figure 4.1 illustrates the hump-shaped pattern of the labor income over the life-cycle. In the main analysis, retirement incomes are assume to be zero.

4. **Other Parameters**. We consider an investor with risk aversion parameter $\gamma = 5$, and a time preference rate $\delta = 0.02$. She starts at age 20 and has 45 years to retirement and 20 years in retirement. Initial nominal wealth, initial nominal labor income and initial price level are set to $W_0 = \$5000$, $L_0 = \$15000$ and $\Pi_0 = \$1$, respectively. In the benchmark case, we ignore the retirement income. Without loss of generality, we fix y_0, r_0 and π_0 at their steady state levels, namely, 0, $\bar{r}$ and $\bar{\pi}$. The maturities of the two nominal bonds are chosen to be 1-year and 10-year and that of the inflation-indexed bond is 1-year. All of these parameter values are summarized in Table 4.3.

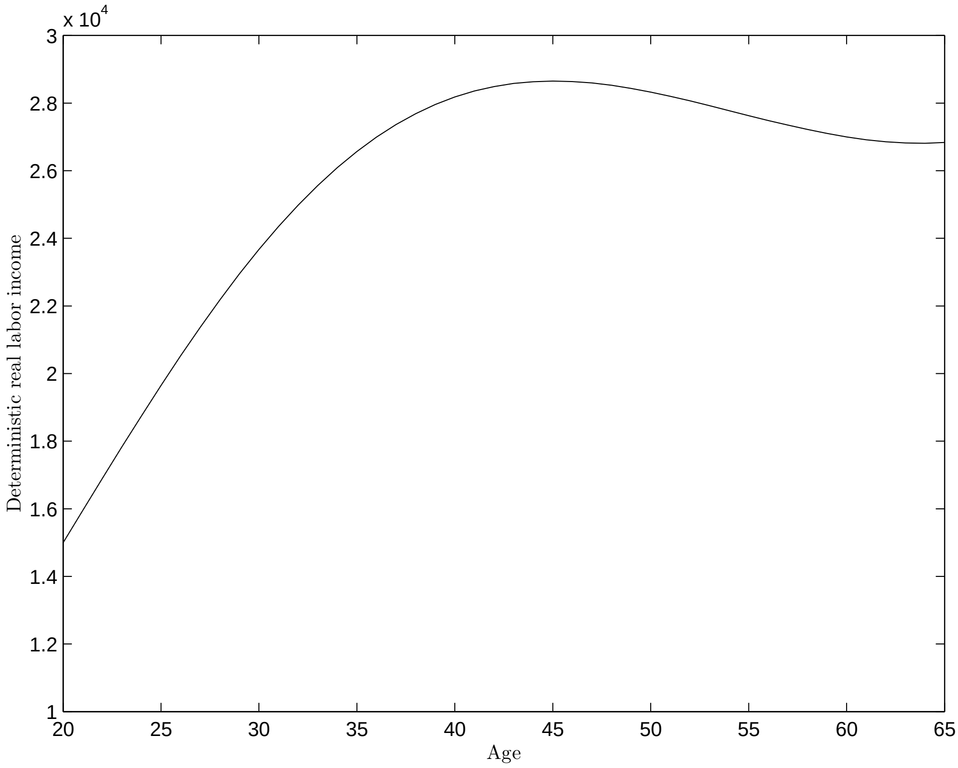


Figure 4.1: Real labor income over the life-cycle. The figures shows how real labor income of the agent evolves over the life-cycle. Her initial real labor income is 15000 and she receives income streams until retirement at age of 65. Other parameter values are: $a_0 = 0.1682$, $a_1 = -0.00646$, $a_2 = 0.00006$.

4.4 Explicit Solution without Income Risks and Investment Constraints

In this section we analytically solve the utility optimization problem outlined above in the absence of both unspanned income risks and investment constraints. We therefore assume $\psi_y = 0$ and $\psi_l = 0$ so that the income streams are only exposed to financial market risks. Although these assumptions make the setting unrealistic for most households, they ensure the existence of explicit solution, which facilitates an understanding and a quantification of the economic forces at play and allows us to derive relevant implications. Hence, we start our analysis by investigating this analytically tractable setting. First, we derive and discuss the solution. Then, we provide numerical illustrations of the solution using the calibrations sketched in Section 4.3. Specifically, we analyze how the risk characteristics of human capital vary with the strength of cointegration effect

Table 4.2: Calibrated parameter values for asset price

Parameter	Value
Stock Return Process: $\mathrm{d}S/S = (R + \lambda_S\sigma_S)\mathrm{d}t + \sigma_S\mathrm{d}z_S$	
σ_S	0.1640
λ_S	0.3269
Real Interest Rate Process: $\mathrm{d}r = \kappa_r(\bar{r} - r)\mathrm{d}t + \sigma_r\mathrm{d}z_r$	
$\bar{r}$	0.0095
κ_r	0.4148
σ_r	0.0181
λ_r	-0.2155
Expected Inflation Process: $\mathrm{d}\pi = \kappa_\pi(\bar{\pi} - \pi)\mathrm{d}t + \sigma_\pi\mathrm{d}z_\pi$	
$\bar{\pi}$	0.0387
κ_π	0.0544
σ_π	0.0171
λ_π	-0.1023
Realized Inflation Process: $\frac{\mathrm{d}\Pi}{\Pi} = \pi\mathrm{d}t + \xi_S\mathrm{d}z_S + \xi_r\mathrm{d}z_r + \xi_\pi\mathrm{d}z_\pi + \xi_u\mathrm{d}z_u$	
ξ_S	0.0019
ξ_r	0.0077
ξ_π	0.0106
ξ_u	0.0182
λ_u	0.0182
Correlations	
ρ_{Sr}	0.0749
$\rho_{S\pi}$	-0.2490
$\rho_{r\pi}$	-0.5355

This table presents the calibrated parameter values for asset price. The parameters for the stocks are calibrated to the value-weighted returns on an index comprising all NYSE, AMEX and NASDAQ firms for the same sample period. All parameter values are annualized. The parameters for the real interest, expected inflation and unexpected inflation rate are calibrated to quarterly U.S. data on yields of five constant maturity Treasury bonds and inflation for the period from the second quarter of 1952 to the fourth quarter of 2012.

Table 4.3: Choice of other parameters

Parameter	Value
Idiosyncratic Labor Income Process: $\frac{\mathrm{d}L_{2t}}{L_{2t}} = (a_0 + a_1 t + a_2 t^2)\mathrm{d}t + \psi_l \mathrm{d}z_{lt}$	
a_0	0.1682
a_1	-0.00646
a_2	0.00006
ψ_l	0.1
Aggregate Labor Income Process:	
κ_y	0.005
ψ_S	0.0019
ψ_r	0.0077
ψ_π	0.005
ψ_u	0.01
ψ_y	0.02
Bond Maturities	
s_1	1
s_2	10
s_3	10
Preference Parameters	
Risk aversion (γ)	5
Subjective discount rate (δ)	0.02
Initial Conditions	
Initial price level (Π_0)	\$1
Initial nominal wealth (W_0)	\$5000
Initial nominal labor income (L_0)	\$15000

The table presents the parameters values that need to be set in addition to the calibrated parameters for asset price. The parameter values for the idiosyncratic labor income dynamics are taken from Cocco, Gomes, and Maenhout (2005) and represent values for high school group.

and degree of wage rigidity and discuss how this variation affects the optimal portfolio strategy over the life-cycle.

4.4.1 The Solution

We first specify the asset menu of the economy. Equation (4.10) shows that nominal bonds have loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$, but no loading on $\mathrm{d}z_u$. Therefore, in an economy without inflation-indexed bonds, the residual inflation risk can not be spanned and the market is incomplete. This corresponds to the setting of Brennan and Xia (2002), who show that in the absence of labor income, finding the optimal investment strategy in closed form is possible in such an incomplete market, because financial wealth can be decomposed as the product of a hedgeable component and an independent component that is driven by unexpected inflation shocks and can be interpreted as exogenous background risk. As a consequence, unspanned risks matter for utility but not for trading strategy. Nonetheless, the presence of labor income destroys the multiplicative nature of the unhedgeable risk and makes the decomposition impossible unless nominal labor income has no exposure to the unexpected inflation risk.[9] This is due to the fact that in the presence of labor income, households trade financial assets to replicate not the optimal risk exposure of financial wealth but rather that of total wealth, which consists of both financial wealth and human capital. However, in general the two types of wealth respond to unexpected inflation shocks differently, which rules out the possibility of decomposing total wealth into two independent components. Therefore, we include an inflation-indexed bond in the asset menu to complete the financial market, because, as shown in (4.14), inflation-indexed bonds have non-zero loading on $\mathrm{d}z_u$, which allows the investor to hedge against unexpected inflation risk.

Specifically, we assume that the investor can invest in five securities: a nominal riskless asset, a stock, two nominal bonds with different maturities T_1 and T_2 and an inflation-indexed bond with maturity T_3. Let σ be the factor loadings matrix of the stock and three bonds and Λ be the vector of the nominal risk premia, which are given

[9] Proof of this impossibility is available upon request.

by,

$$\sigma = \begin{pmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & -B_r(T_1)\sigma_r & -B_\pi(T_1)\sigma_\pi & 0 \\ 0 & -B_r(T_2)\sigma_r & -B_\pi(T_2)\sigma_\pi & 0 \\ \xi_S & \xi_r - B_r(T_3)\sigma_r & \xi_\pi & \xi_u \end{pmatrix}, \tag{4.24}$$

and

$$\begin{aligned}\Lambda = \sigma\lambda = (&\sigma_S\lambda_S, -B_r(T_1)\sigma_r\lambda_r - B_\pi(T_1)\sigma_\pi\lambda_\pi, -B_r(T_2)\sigma_r\lambda_r - B_\pi(T_2)\sigma_\pi\lambda_\pi, \\ &-B_r(T_3)\sigma_r\lambda_r + \xi'\lambda)'.\end{aligned} \tag{4.25}$$

The nominal wealth dynamics can be written as,

$$\mathrm{d}W_t = [W_t\,(R_t + x_t'\Lambda) - C_t + L_t]\,\mathrm{d}t + x_t'\sigma W_t \mathrm{d}z_t. \tag{4.26}$$

where x is the vector of the fractions of financial wealth invested in the risky assets. The investor maximizes her utility by appropriately choosing a nominal consumption process $C = (C_t)$ and a portfolio strategy $x = (x_t)$. Define the indirect utility at time t, J_t, as the highest level of utility that can be obtained in the remaining lifetime,

$$J(t, r_t, y_t, L_t, W_t, \Pi_t) = \max_{(C,x)\in A} \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\delta(v-t)}\frac{(\frac{C_v}{\Pi_v})^{1-\gamma}}{1-\gamma}\mathrm{d}v\right]. \tag{4.27}$$

where A is the set of admissible consumption and portfolio strategy over the period $[t, T]$.

In the absence of idiosyncratic labor income shocks and market frictions, labor incomes can be valued as the dividend streams from an artificial asset, which is referred to as human capital. The time t real human capital, h_t, is

$$h_t = \mathrm{E}_t\left[\int_t^T \frac{m_s}{m_t} l_s \mathrm{d}s\right]. \tag{4.28}$$

h_t can be thought of as the amount of wealth the investor can make by selling her future labor income streams. Using the dynamics of y in (4.18) and real labor income dynamics in (4.22), we are able to derive an explicit expression of real human capital as shown in Proposition 1.

Proposition 1. *For $t \leq \hat{T}$, the time t real human capital is*

$$h(t, r_t, y_t, l_t) = l_t D(t, r_t, y_t) = l_t \int_t^{\hat{T}} \mathrm{e}^{F(t,s) - \kappa_y B_y(s-t) y_t} p_t^s \mathrm{d}s, \tag{4.29}$$

where

$$\begin{aligned}
B_y(\tau) &= \kappa_y^{-1}(1 - e^{-\kappa_y \tau}), && (4.30)\\
p_t^s &= p^s(t, r_t) = \exp\left[A(s-t) - B_r(s-t) r_t\right], && (4.31)\\
A(\tau) &= \left(\bar{r} + \frac{\phi' \rho e_2 \sigma_r}{\kappa_r} - \frac{\sigma_r^2}{2\kappa_r^2}\right)[B_r(\tau) - \tau] - \frac{\sigma_r^2}{4\kappa_r} B_r^2(\tau), && (4.32)\\
F(t, s) &= \left[a_0 + \frac{1}{2} a_1(s+t) + \frac{1}{3} a_2(s^2 + st + t^2)\right](s-t) + \frac{(\psi - \xi)' \rho e_2 \sigma_r}{\kappa_r + \kappa_y} B_r(s-t)\\
&\quad + \left((\psi - \xi)' \rho \phi + \frac{(\psi - \xi)' \rho (\psi - \xi)}{2} - \frac{(\psi - \xi)' \rho e_2 \sigma_r}{\kappa_r + \kappa_y}\right) B_y(s-t)\\
&\quad - \frac{(\psi - \xi)' \rho (\psi - \xi) \kappa_y}{4} B_y^2(s-t) - \frac{(\psi - \xi)' \rho e_2 \sigma_r \kappa_y}{\kappa_r + \kappa_y} B_r(s-t) B_y(s-t).\\
& && (4.33)
\end{aligned}$$

The dynamics of real human capital are

$$\frac{\mathrm{d}h_t}{h_t} = \mathrm{d}BV + \left[b_{1t}(\psi - \xi) - b_{2t}\sigma_r e_2\right]' \mathrm{d}z_t, \tag{4.34}$$

where

$$\begin{aligned}
b_{1t} &= 1 - \frac{\kappa_y l_t}{h_t} \int_t^{\hat{T}} B_y(s-t) \mathrm{e}^{F(t,s) - \kappa_y B_y(s-t) y_t} p(s-t) \mathrm{d}s, && (4.35)\\
b_{2t} &= \frac{l_t}{h_t} \int_t^{\hat{T}} B_r(s-t) \mathrm{e}^{F(t,s) - \kappa_y B_y(s-t) y_t} p(s-t) \mathrm{d}s, && (4.36)
\end{aligned}$$

and BV is short for some bounded variation process and can be different in each occurrence of abbreviation.

Proposition 1 reveals that the agent's real human capital is decreasing in age and is a product of the current labor income l, and a multiplier, which depends on time, real interest rate r and cointegration variable y. In the absence of cointegration, which corresponds to $\kappa_y = 0$, the multiplier can be interpreted as the time t market price of a bond paying deterministic real coupons. In contrast, the presence of cointegration makes coupons stochastic and alters the risk characteristics of human capital. To better illustrate this effect, we derive the dynamics of nominal human capital using the dynamics of real human capital and realized inflation:

$$
\begin{aligned}
\frac{\mathrm{d}H_t}{H_t} &= \mathrm{d}BV + (\xi + b_{1t}(\psi - \xi) - b_{2t}\sigma_r e_2)'\mathrm{d}z_t, \\
&= \mathrm{d}BV + [b_{1t}\psi + (1 - b_{1t})\,\xi - b_{2t}\sigma_r e_2]'\,\mathrm{d}z_t. \qquad (4.37)
\end{aligned}
$$

Obviously, the risk exposures of nominal human capital contain three components. The first two capture the inflation risk exposures of nominal human capital: they are weighted averages of the risk exposures of nominal labor income, ψ, and those of realized inflation, ξ. The weight is b_{1t}, which can be shown to fall between zero and one. Since $\psi_\pi \leq \xi_\pi$ and $\psi_u \leq \xi_u$, the smaller b_{1t} is, the stronger the inflation hedge provided by human capital. In contrast to the no cointegration case of $b_{1t} = 1$, the dependence on y induced by cointegration makes b_{1t} smaller than one and thus enhances the inflation hedging power of human capital. The economic intuition is that the reductions in value of nominal human capital caused by instantaneous inflation shocks is compensated by opposite movements in the growth rate of labor income induced by the long-run dependence. As a consequence, the effect of inflation erosion on human capital is mitigated. In sum, the inflation hedging property of human capital depends on both the short-term correlation between labor income and inflation captured by ψ and the long-term correlation between human capital and inflation captured by b_{1t}. The last component $b_{2t}\sigma_r e_2$ captures the interest rate risk exposure of human capital caused by the variation in the discount rate. Obviously, it decreases with age because of the depletion of human capital.

Under the complete market assumption, the budget constraint can be reformulated as,

$$
\mathrm{E}\left[\int_0^T \frac{m_t}{m_0}\frac{C_t}{\Pi_t}\mathrm{d}t\right] \leq \frac{W_0}{\Pi_0} + \mathrm{E}\left[\int_0^T \frac{m_t}{m_0}\frac{L_t}{\Pi_t}\mathrm{d}t\right]. \qquad (4.38)
$$

where W_0 is the nominal initial wealth, Π_0 is the initial price level. Equation (4.38) implies that future consumption streams must be financeable by the sum of the initial wealth and future labor income streams. Following Brennan and Xia (2002), we can solve this optimization problem using the martingale approach. Note that in the optimization problem with labor income, the optimal strategy ensures that total wealth stays positive with certainty. This implies, however, financial wealth is allowed to go negative. For such negative values of financial wealth, it makes little sense to talk of the fraction of financial wealth invested. Therefore, we instead characterize the optimal portfolio strategy in terms of portfolio weights in total wealth, θ.[10] Proposition 2 summarizes the solution.

Proposition 2. *The optimal real consumption rate is*

$$c_t^* = Q^{-1}(t, r_t)\left[w_t^* + h(t, r_t, y_t, l_t)\right], \tag{4.39}$$

where $Q(t, r_t)$ is defined by

$$Q(t, r_t) = \int_t^T \exp\left\{\frac{1-\gamma}{\gamma}\left[B_r(t-s)r_t + \alpha(t-s)\right]\right\} \mathrm{d}s \tag{4.40}$$

with $\alpha(t, s)$ defined by

$$\begin{aligned}\alpha(\tau) = &\left[\bar{r} + \frac{1-\gamma}{\gamma}\left(\frac{\sigma_r^2}{2\kappa_r^2} - \frac{\phi'\rho e_2 \sigma_r}{\kappa_r}\right)\right]\left[\tau - B_r(\tau)\right] \\ &+ \left(\frac{\delta}{\gamma - 1} + \frac{\phi'\rho\phi}{2\gamma}\right)\tau - \frac{(1-\gamma)\sigma_r^2}{4\gamma\kappa_r} B_r^2(\tau).\end{aligned} \tag{4.41}$$

The indirect utility function is

$$J(t, r_t, y_t, L_t, W_t, \Pi_t) = Q^\gamma(t, r_t)\frac{\left[w_t^* + h(t, r_t, y_t, l_t)\right]^{1-\gamma}}{1-\gamma}. \tag{4.42}$$

The risk exposures of the optimal nominal financial wealth, $K_t^* = (K_{St}^*, K_{rt}^*, K_{\pi t}^*, K_{ut}^*)'$, are

$$\begin{aligned}K_t^* = &\frac{1}{\gamma}(\sigma_{vol})^{-1}(-\phi) + \frac{1-\gamma}{\gamma}\bar{B}_r(t, r_t)e_2 \\ &+ (\sigma_{vol})^{-1}\xi - (\sigma_{vol})^{-1}\left(\xi + b_{1t}(\psi - \xi) - b_{2t}\sigma_r e_2\right)\frac{h_t}{w_t^* + h_t},\end{aligned} \tag{4.43}$$

where

$$\bar{B}_r(t, r_t) = Q^{-1}(t, r_t)\int_t^T \exp\left\{\frac{1-\gamma}{\gamma}\left[B_r(s-t)r_t + \alpha(s-t)\right]\right\} B_r(s-t)\mathrm{d}s, \tag{4.44}$$

$\sigma_{vol} = \operatorname{diag}(\sigma_S, \sigma_r, \sigma_\pi, \xi_u)$ *and w_t^* is the real wealth process induced by the optimal strat-*

[10] It is easy to verify that the relationship between x_t and θ_t is given by $\theta_t = (x_t h_t)/(w_t + h_t)$.

egy.

The optimal portfolio weights in total wealth, $\theta_t^* = (\theta_{St}^*, \theta_{N_1t}^*, \theta_{N_2t}^*, \theta_{It}^*)'$, are

$$
\begin{aligned}
\theta_t^* &= \frac{1}{\gamma}(\sigma')^{-1}(-\phi) + \frac{1-\gamma}{\gamma}\bar{B}_r(t, r_t)(\sigma')^{-1}\sigma_r e_2 \\
&\quad + (\sigma')^{-1}\xi - (\sigma')^{-1}\left(\xi + b_{1t}(\psi - \xi) - b_{2t}\sigma_r e_2\right)\frac{h_t}{w_t^* + h_t} \\
&= \frac{1}{\gamma}\Sigma^{-1}\Lambda + \left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\sigma\rho\left[\xi - \bar{B}_r(t, r_t)\sigma_r e_2\right] \\
&\quad - \Sigma^{-1}\sigma\rho\left[b_{1t}\psi + (1 - b_{1t})\,\xi - b_{2t}\sigma_r e_2\right]\frac{h_t}{w_t^* + h_t},
\end{aligned}
\tag{4.45}
$$

where $\Sigma = \sigma\rho\sigma'$ is the variance-covariance matrix of the nominal asset returns.

As shown in equation (4.39), the optimal consumption is a time and real interest rate-dependent fraction of total wealth. It can easily be verified that the propensity to consume out of financial wealth and the propensity to consume out of current income, are given by,

$$
\frac{c_t}{w_t} = Q^{-1}(t, r_t)\left[1 + \frac{l_t}{w_t}D(t, r_t, y_t)\right], \qquad \frac{c_t}{l_t} = Q^{-1}(t, r_t)\left[\frac{w_t}{l_t} + D(t, r_t, y_t)\right]. \tag{4.46}
$$

Obviously, both measures depend on the wealth-income ratio.

Equation (4.43) expresses the optimal portfolio strategy in terms of risk exposures. As shown in Appendix 4.6.3, in deriving the solution, we first obtain the risk exposures of the optimal financial wealth and then determine the optimal portfolio weights depending on the asset menu. Put differently, the optimal portfolio weights are dependent on the choice of assets, such as the maturities of the bonds, but the risk exposures are not. Therefore, although they are interchangeable in terms of characterizing the optimal portfolio strategy, we will primarily focus on the optimal risk exposures. Note that σ_{vol} contains the volatilities associated with the four risk factors and serves as a scaling factor.

Equation (4.45) expresses the optimal portfolio as the sum of three components: a speculative portfolio that invests in the nominal mean-variance tangency portfolio represented by $\Sigma^{-1}\Lambda$, an intertemporal hedge portfolio that hedges against adverse variation of future investment opportunities, and a correction portfolio that corrects for the implicit investment through nominal human capital. Comparing with Brennan and Xia

(2002), we see that the presence of labor income induces a new correction portfolio that is independent of the investor's risk attitude. Under the assumptions of no labor income risks and no investment constraints, the optimization problem with an initial real financial wealth of w_t and labor income is equivalent to that with an initial real financial wealth of $w_t + h_t$ and no labor income. The investor first determines her optimal investment of total wealth and then corrects it for the implicit investment, which crucially depends on the riskiness of human capital. The effect of cointegration on the correction portfolio is parameter specific. As discussed in Section 4.3, it is reasonable to assume $\psi_\pi \leq \xi_\pi$ and $\psi_u \leq \xi_u$. Under these assumptions, the cointegration between labor income and inflation expands the correction portfolio, because it reduces b_{1t}. As a consequence, human capital provides larger inflation hedge for itself and less inflation hedge through financial assets is needed. Moreover, in general, we have $\partial d_{1t}/\partial \kappa_y < 0$, which implies that the faster the mean reversion of y, the larger the increase in the correction portfolio and the stronger the effect of cointegration on the optimal portfolio strategy.

4.4.2 Numerical Illustrations

In this section, we carry out some numerical experiments to illustrate the effects of cointegration on the riskiness of human capital and consequently on the optimal portfolio strategy. Specifically, we consider two distinct cases: partial wage rigidity where the contemporaneous correlation between nominal labor income and inflation is positive, and nominal wage rigidity where the contemporaneous correlation is nearly zero. The benchmark asset menu consists of a stock, a 1-year nominal bond, a 10-year nominal bond, an 1-year inflation-indexed bond with maturity and cash. As the optimal portfolio weights depend on the risk characteristics of the assets, we also study the case in which the 1-year inflation-indexed bond is replaced by a 10-year inflation-indexed bond and see how this change affects the optimal portfolio weights.

Decomposition of Nominal Human Capital

As explained above, in a complete market without unspanned income risks and investment constraints, nominal human capital can be spanned by the traded assets. Since the factor loadings matrix σ is defined in terms of volatilities of nominal asset returns, we consider a portfolio that replicates the long position in nominal human capital. Using

Table 4.4: Decomposition of Nominal Human Capital

	(a) Partial Wage Rigidity							
	Risk Exposures				Portfolio Weights			
	K_S^H	K_r^H	K_π^H	K_u^H	θ_S^H	$\theta_{N_1}^H$	$\theta_{N_2}^H$	θ_I^H
$\kappa_y = 0$	0.011	-1.859	0.327	0.451	0.006	3.268	-0.419	0.451
$\kappa_y = 0.05$	0.011	-1.859	0.498	0.770	0.003	3.011	-0.383	0.770
$\kappa_y = 0.1$	0.011	-1.860	0.552	0.872	0.001	2.929	-0.371	0.872

	(b) Nominal Wage Rigidity							
	Risk Exposures				Portfolio Weights			
	K_S^H	K_r^H	K_π^H	K_u^H	θ_S^H	$\theta_{N_1}^H$	$\theta_{N_2}^H$	θ_I^H
$\kappa_y = 0$	0.011	-1.857	0	0	0.011	3.578	-0.452	0
$\kappa_y = 0.05$	0.011	-1.859	0.360	0.581	0.005	3.141	-0.396	0.581
$\kappa_y = 0.1$	0.011	-1.859	0.475	0.767	0.003	3.000	-0.379	0.767

The table shows the decomposition of a portfolio that replicates the long position in nominal human capital at age of 20. Two types of wage rigidity are considered: partial wage rigidity ($\psi_\pi = 0.005$ and $\psi_u = 0.01$) and nominal wage rigidity ($\psi_\pi = 0$ and $\psi_u = 0$). κ_y is the mean reversion coefficient for the cointegration variable y. K_S^h, K_r^h, K_π^h and K_u^h are the loadings of the replicating portfolio on the innovations in z_S, z_r, z_π and z_u, respectively. θ_S^h, $\theta_{N_1}^h$, $\theta_{N_2}^h$ and θ_I^h are the portfolio weights to the stock, the 1-year nominal bond, the 10-year nominal bond and the 1-year inflation-indexed bond, respectively.

the dynamics of nominal human capital in (4.37), we can compute the factor loadings, $K_t^H = (K_{St}^H, K_{rt}^H, K_{\pi t}^H, K_{ut}^H)'$, and the weights, $\theta_t^H = (\theta_S^H, \theta_{N_1 t}^H, \theta_{N_2 t}^H, \theta_{It}^H)'$, of the replicating portfolio:

$$K_t^H = \sigma_{vol}^{-1} \left[b_{1t}\psi + (1 - b_{1t})\xi - b_{2t}\sigma_r e_2 \right], \tag{4.47}$$

$$\theta_t^H = (\sigma')^{-1} \left[b_{1t}\psi + (1 - b_{1t})\xi - b_{2t}\sigma_r e_2 \right]. \tag{4.48}$$

Table 4.4 illustrates the decomposition of the replicating portfolio for nominal human capital into factor loadings and asset holdings at age of 20. Under partial wage rigidity (Panel (a)), the expected and unexpected inflation risk exposures are magnified by the stronger cointegration effect associated with larger κ_y, which implies that the long-run cointegration makes nominal human capital covary more closely with the inflation realization and therefore mitigates the negative impact of inflation shocks. This can be explained by the fact that instantaneous inflation shocks to the current labor income are offset by increases in expected labor income growth rate induced by the long-run relationship between labor income and inflation and this effect is more pronounced for stronger long-run dependence. The assumption of $\psi_S = \xi_S$ implies that the equity loading of nominal human capital is solely from that of the realized inflation (ξ_S). Therefore, it is rather small and does not vary with κ_y. By contrast, the real interest rate exposure of nominal human capital is large in absolute value because the stochasticity in the discount rate exposes all future labor income streams to real interest rate risk. Although the interest rate loading of nominal human capital is not subject to the cointegration effect either, it exhibits little variation with κ_y. This is due to the minor effect of κ_y on the valuation of human capital.

Under nominal wage rigidity (Panel (b)), nominal labor income has no loadings on inflation shocks and resembles cash stream.[11] When there is no cointegration ($\kappa_y = 0$), nominal wage rigidity gives rise to the insensitivity of nominal labor income streams to inflation shocks. The replicating portfolio has zero exposures to both expected and unexpected inflation risks. In contrast, in the presence of cointegration, instantaneous inflation shocks have long-run effects: while they erode current real labor income, they drive up the expected growth rate for future income streams through cointegration. Therefore, the inflation exposures of nominal human capital are increased. This observation suggests that even if the instantaneous labor income provides no inflation protection due to low contemporaneous correlation between labor income and inflation, the high long-run correlation caused by cointegration makes the entire human capital a solid shield against inflation shocks.

As to the implicit asset holdings, since the unexpected inflation shocks can only be spanned by the inflation-indexed bond, the rise in the unexpected inflation exposure leads to increasing portfolio weight to the inflation-indexed bond. Although the equity exposure stays constant, the implicit holding of the stock declines because of the small and positive correlation between the stock and the inflation-indexed bond. Because the interest rate exposure of nominal human capital stays almost constant and that provided by inflation-indexed bond increases with κ_y, the absolute interest rate exposure left to be

[11] Note that nominal labor income does not reduce to cash stream under nominal wage rigidity. The reason is that the equity exposure and interest rate exposure of real cash stream are opposite to those of inflation realization, but those of nominal labor income are not due to the assumption of $\psi_S = \xi_S$, $\psi_r = \xi_r$.

offered by nominal bonds declines. Thus, the absolute holdings of both nominal bonds drops when the cointegration relationship intensifies.

Figure 4.2 shows the risk exposures of nominal human capital over the life-cycle in the case of partial wage rigidity. While the expected and unexpected inflation exposures stay constant in the absence of cointegration, they are much higher when the investor is young and decrease over time in the presence of ointegration. This is due to the fact that as the agent approaches retirement, the long-run correlation induced by cointegration becomes weaker and irrelevant. The drop in the absolute interest rate exposure can be attributed to the decline in the investment horizon. At the end of the working phase, the interest rate exposure even goes positive since the discount effect almost disappears. The equity exposure hardly exhibits life-cycle variation and is close to zero because of the low equity exposure of labor income streams.

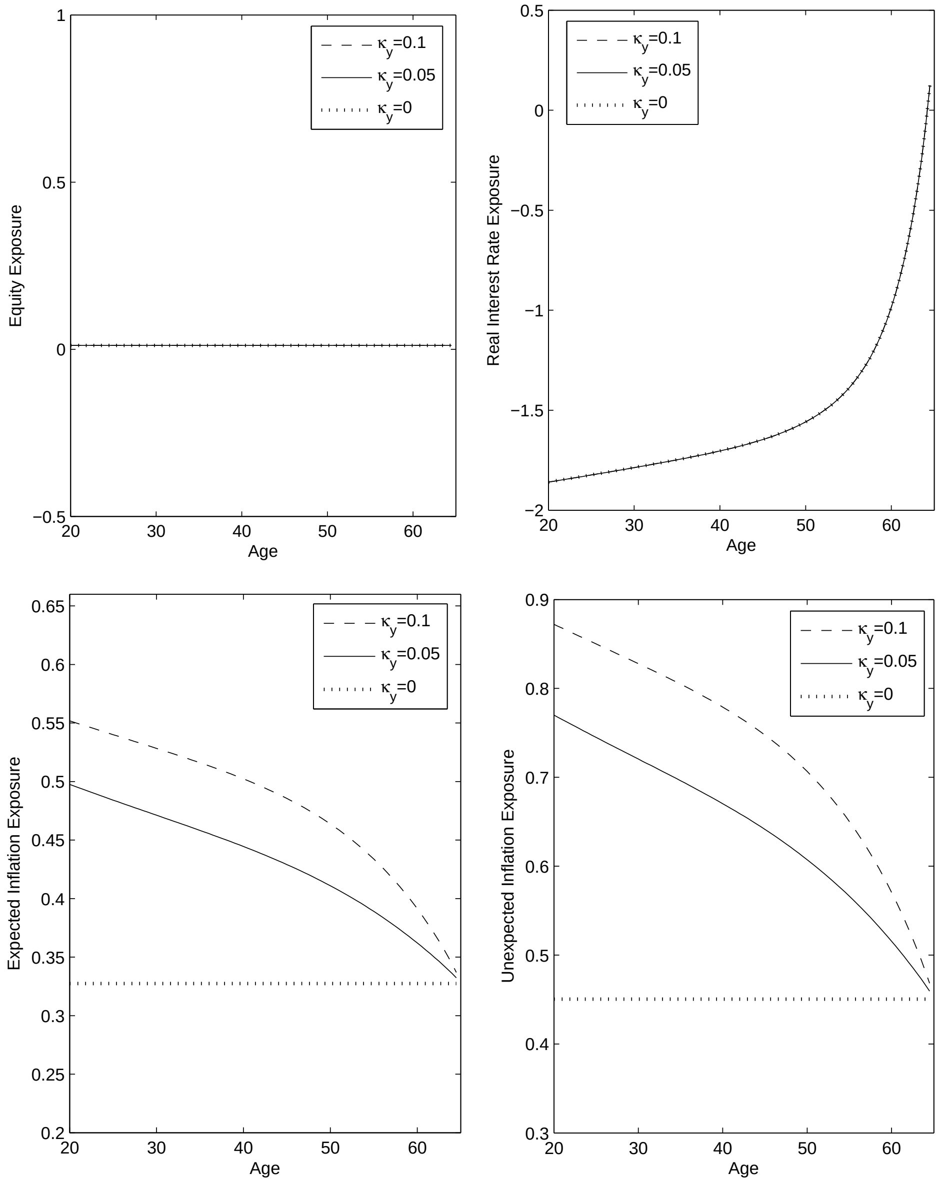


Figure 4.2: Decomposition of nominal human capital over the life-cycle in terms of risk exposures in the case of partial wage rigidity ($\psi_\pi = 0.005$ and $\psi_u = 0.01$) . The figure shows how the equity exposure (top-left), the real interest rate exposure (top-right), the expected inflation risk exposure (bottom-left) and the unexpected inflation risk exposure (bottom-right) of a portfolio that replicates the long position in nominal human capital evolve over the life-cycle in the case of partial wage rigidity where $\psi_\pi = 0.005$ and $\psi_u = 0.01$. The dotted curves depict the risk exposures when $\kappa_y = 0$. The solid curves depict the risk exposures when $\kappa_y = 0.05$. The dashed curves depict the risk exposures when $\kappa_y = 0.1$. The graphs are drawn by taking averages over 1000 simulated paths using optimal strategies.

Figure 4.3 illustrates the implicit holdings of nominal human capital. The replicating portfolio weights in the stock and the inflation-indexed bond mirror the equity exposure and unexpected inflation exposure, because the unexpected inflation exposure can only be obtained by holding the inflation-indexed bond and the equity exposure is mostly offered by the stock.[12] The sharp decline in the interest rate exposure over the life-cycle drives down the absolute holdings of the two nominal bonds. The results in the case of nominal wage rigidity display similar life-cycle variations, which are not shown here in order to save space.

The Optimal Portfolio Strategy

Now we study the optimal portfolio strategy over the life-cycle. Figure 4.4 illustrates how the risk exposures of the financial wealth evolve over the life-cycle in the benchmark case of partial wage rigidity. It is important to realize that the aging of the investor alters two key variables. First, the investment horizon shortens. Second, the composition of total wealth varies, because for a typical investor the ratio of human capital to total wealth is quite high early in life and decreases to zero as retirement approaches.

In the bottom-right panel, we see that the optimal unexpected inflation exposure is increasing in age. The assumption of $\phi_u = 0$ implies that there is no speculative demand for the unexpected inflation risk exposure. Hence, the unexpected inflation risk exposure is fully determined by inflation hedge motive. As the inflation-indexed bond is the only asset available that enables the investor to hedge against the unexpected inflation shocks, this finding reveals that the inflation-indexed bond is less important for young investors than for their older counterpart in terms of hedging unexpected inflation shocks. This distinction stems from the fact that when the investor is young, she receives strong protection against unexpected inflation risk from her future nominal labor income, because they have a positive correlation with the inflation realization. As the investor ages, her remaining labor income streams decline and her self-protection weakens. As a consequence, she has to hold more inflation-indexed bond to maintain a full unexpected inflation hedge for her total wealth. It is worth noting that the cointegration between labor income and inflation makes this protection much stronger and this intensifying effect is bigger for faster mean reversion of κ_y. This is because in the presence of long-

[12] Although inflation-indexed bond bears unexpected inflation exposure as well, its bearing is negligibly small.

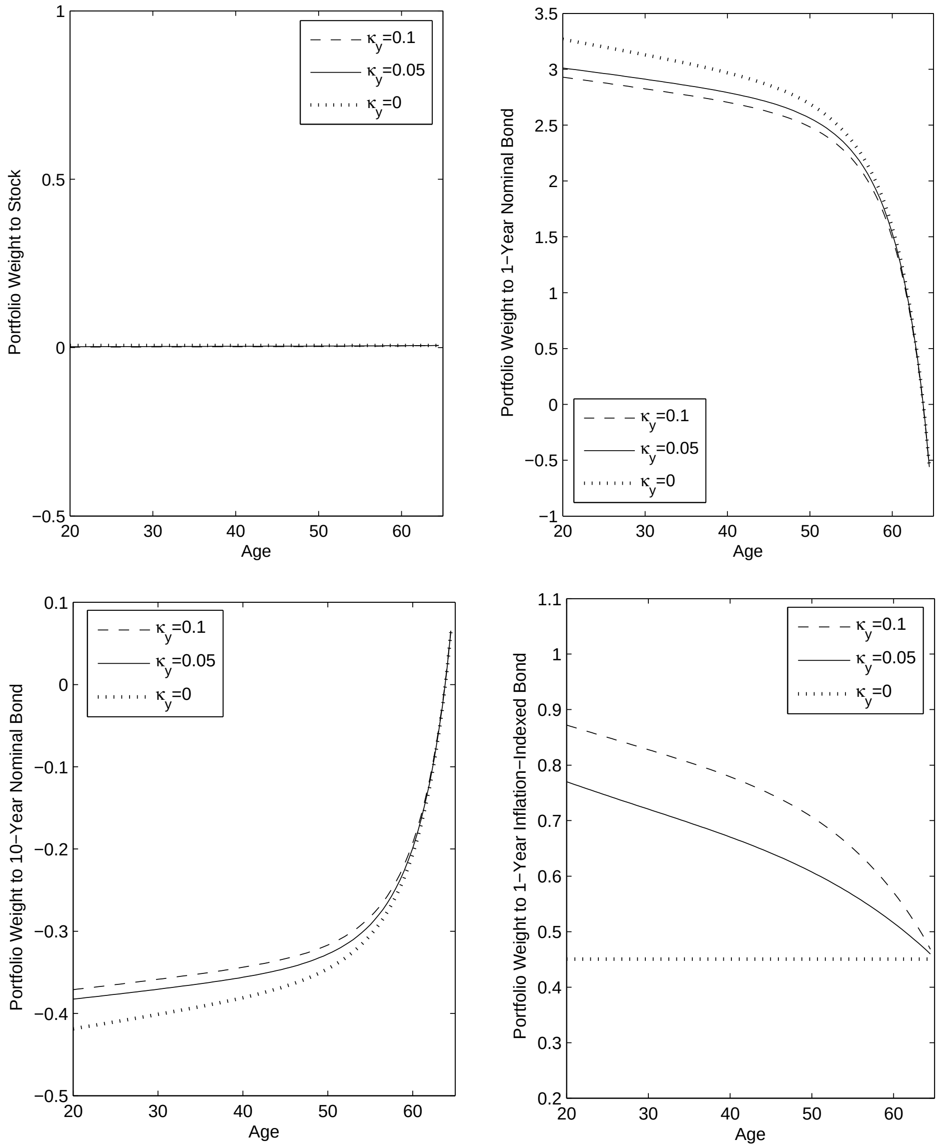


Figure 4.3: Decomposition of nominal human capital over the life-cycle in terms of portfolio weights in the case of partial wage rigidity ($\psi_\pi = 0.005$ and $\psi_u = 0.01$) . The figure shows how the fractions of total wealth invested in the stock (top-left), the 1-year nominal bond (top-right), the 10-year nominal bond (bottom-left) and the 1-year inflation-indexed bond (bottom-right) of a portfolio that replicates the long position in nominal human capital evolve over the life-cycle in the case of partial wage rigidity where $\psi_\pi = 0.005$ and $\psi_u = 0.01$. The dotted curves depict the portfolio weights when $\kappa_y = 0$. The solid curves depict the portfolio weights when $\kappa_y = 0.05$. The dashed curves depict the portfolio weights when $\kappa_y = 0.1$. The graphs are drawn by taking averages over 1000 simulated paths using optimal strategies.

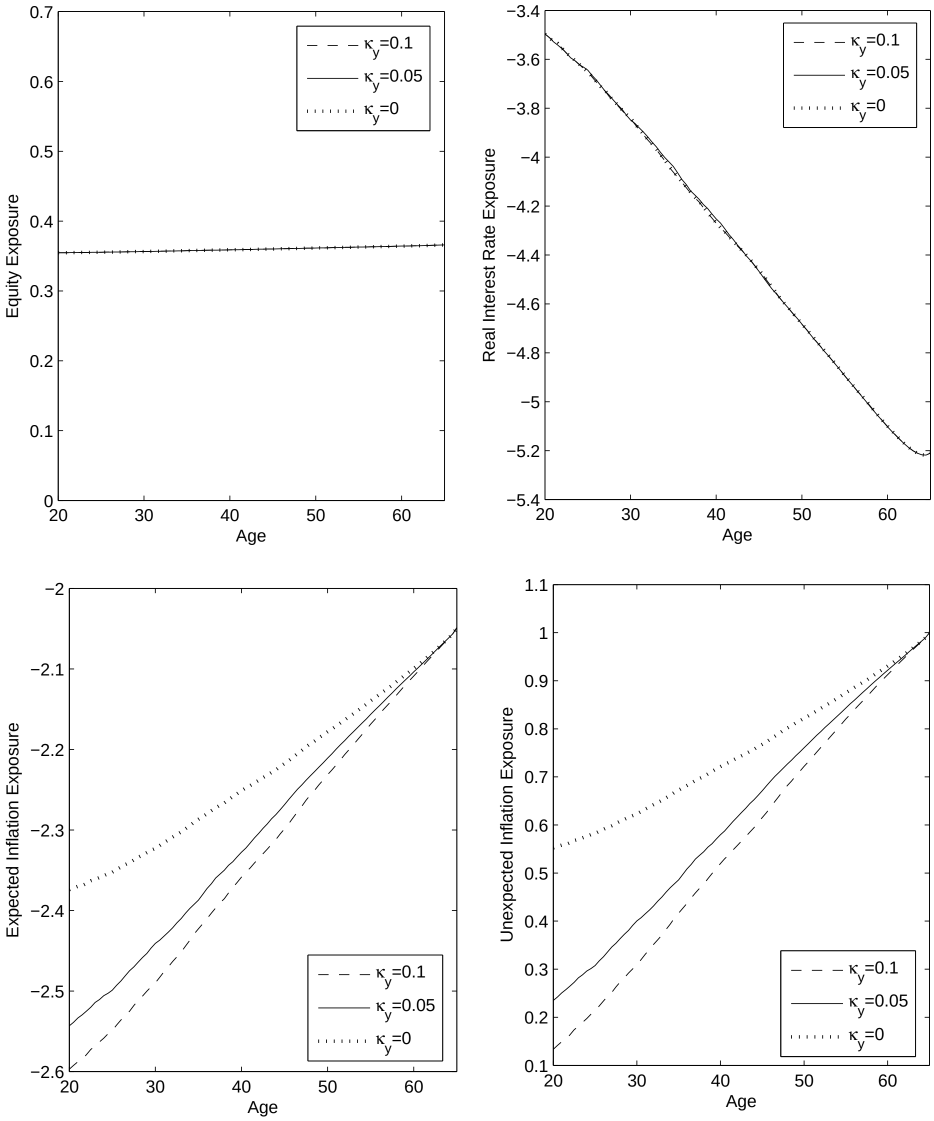


Figure 4.4: Optimal portfolio weights over the life-cycle in the case of partial wage rigidity. The figure shows how the optimal fractions of total wealth invested in the stock (top-left), the 1-year nominal bond (top-right), the 10-year nominal bond (bottom-left) and the 1-year inflation-indexed bond (bottom-right) evolve over the life-cycle in the case of partial wage rigidity ($\psi_\pi = 0.005$ and $\psi_u = 0.01$) .

run dependence, inflation shocks are offset not only by instantaneous compensation from current labor income but also by higher growth in future income streams. This effect diminishes as the composition of the investor's total wealth leans more towards financial wealth. When she exhausts her human capital at retirement, her unexpected inflation risk exposure reaches one, which implies that she completely relies on the inflation-indexed bond to protect her wealth. In the meantime, the curves representing different strength of cointegration converge.

The bottom-left panel of Figure 4.4 shows that the absolute expected inflation exposure declines over the life-cycle. Unlike the unexpected inflation risk, the expected inflation risk is carried by the speculative portfolio. Since its market price is calibrated to be negative as shown in Table 4.2, the investor is inclined to exploit its risk premium by taking negative exposure in the speculative portfolio. On the other hand, the positive correlation between labor income and inflation leaves nominal human capital positively exposed to expected inflation shocks. The correction for this positive exposure invites even higher absolute expected inflation loadings. As the investor ages, the ratio of human capital to total wealth falls and the correction portfolio shrinks. This accounts for the decrease in the absolute expected inflation exposure. The cointegration intensifies the correction effect and leads to higher exposure to the expected inflation risk. The variation in the expected inflation exposure over the life-cycle stands a stark contrast to the finding of Brennan and Xia (2002) that the optimal expected inflation exposure is independent of the investment horizon, because it is solely from the speculative portfolio in their study.

As shown in the top-right panel, the absolute interest rate exposure increases with age. This observation seems puzzling and counterintuitive, since shorter horizons are typically associated with less variation in future investment opportunities and lower intertemporal hedge portfolio. However, in the presence of labor income, lower age means not only longer investment horizon, but also greater share of human capital in total wealth. Therefore, this puzzle can be resolved by taking into account the evolution of the interest rate exposure carried by nominal human capital displayed in Figure 4.2: In contrast to the increasing absolute interest rate exposure of financial wealth, that of nominal human capital declines rapidly over the life-cycle. In an unreported result, we show that the interest exposure of total wealth is almost flat which is consistent with the result of Brennan and Xia (2002) that there is limited horizon effect on the optimal interest rate risk exposure (about five years). The intuition is that without taking into account the interest rate exposure implicitly borne by nominal human capital, our optimization problem reduces to the one studies by Brennan and Xia (2002) with a lump sum increase in the initial wealth. Hence, to correct for the decreasing implicit interest rate exposure of nominal human capital, the investor has to increase that of financial wealth as she ages. Finally, the optimal equity exposure stays almost unchanged because it is mostly determined by the speculative demand, which is independent of age.

Figure 4.5 shows the optimal portfolio weights over the life-cycle in the benchmark case of partial wage rigidity. The evolution of the allocation to the inflation-indexed bond coincides with that of the unexpected inflation exposure because the investor can only use the inflation-indexed bond to hedge against that risk. Although the positions in both the stock and inflation-indexed bond offer equity exposure, the calibrated low contemporaneous correlation between the stock and realized inflation makes the equity exposure from the indexed bond holding quantitatively negligible. This accounts for why the equity investment mirrors equity exposure. As the increase in the absolute interest rate exposure is larger than the decrease in the absolute expected inflation exposure, the absolute holdings of both nominal bonds rise.

Now we turn to the case of nominal wage rigidity. The risk exposures of the optimal nominal financial wealth are illustrated in Figure 4.6. In the absence of cointegration, the investor chooses to achieve a full hedge against the unexpected inflation risks by financial investment, which is reflected by the flat curve at the level of one in the bottom right panel. Similarly, the expected inflation exposure does not vary over the life-cycle. This absence of age effect can be attributed to the riskiness of human capital. When there is no long-run relationship between labor income and inflation, instantaneous inflation shocks only have short-term effects on current labor income. Under nominal wage rigidity, it has zero loadings on innovations in the both types of inflation risks and resembles cash stream. Therefore, human capital, which is composed of future quasi-cash streams, does not respond to inflation shocks and needs a full protection from financial investment. Interestingly, the introduction of cointegration reproduces the age effect on the inflation risk taking. This is due to the fact that while the contemporaneous correlation the labor market and realized inflation is low, their long-run correlation, which is characterized by the correlation between nominal human capital and inflation, is amplified by the cointegration, thereby leading to large implicit loadings of nominal human capital on inflation risks. This means that even if labor income streams provide no inflation protection, human capital can still serve as a good inflation hedge. However, as the ratio of human capital to total wealth goes down over time, the importance of implicit loadings fades and more explicit loadings from financial investment are needed. This also explains the constant demand for the indexed bond in the absence of cointegration, but increasing demand for it in the presence of cointegration as shown in Figure 4.7.

Finally, we investigate how the choice of assets affects the optimal portfolio weights. For this purpose, we change the maturity of the inflation-indexed to ten years. The

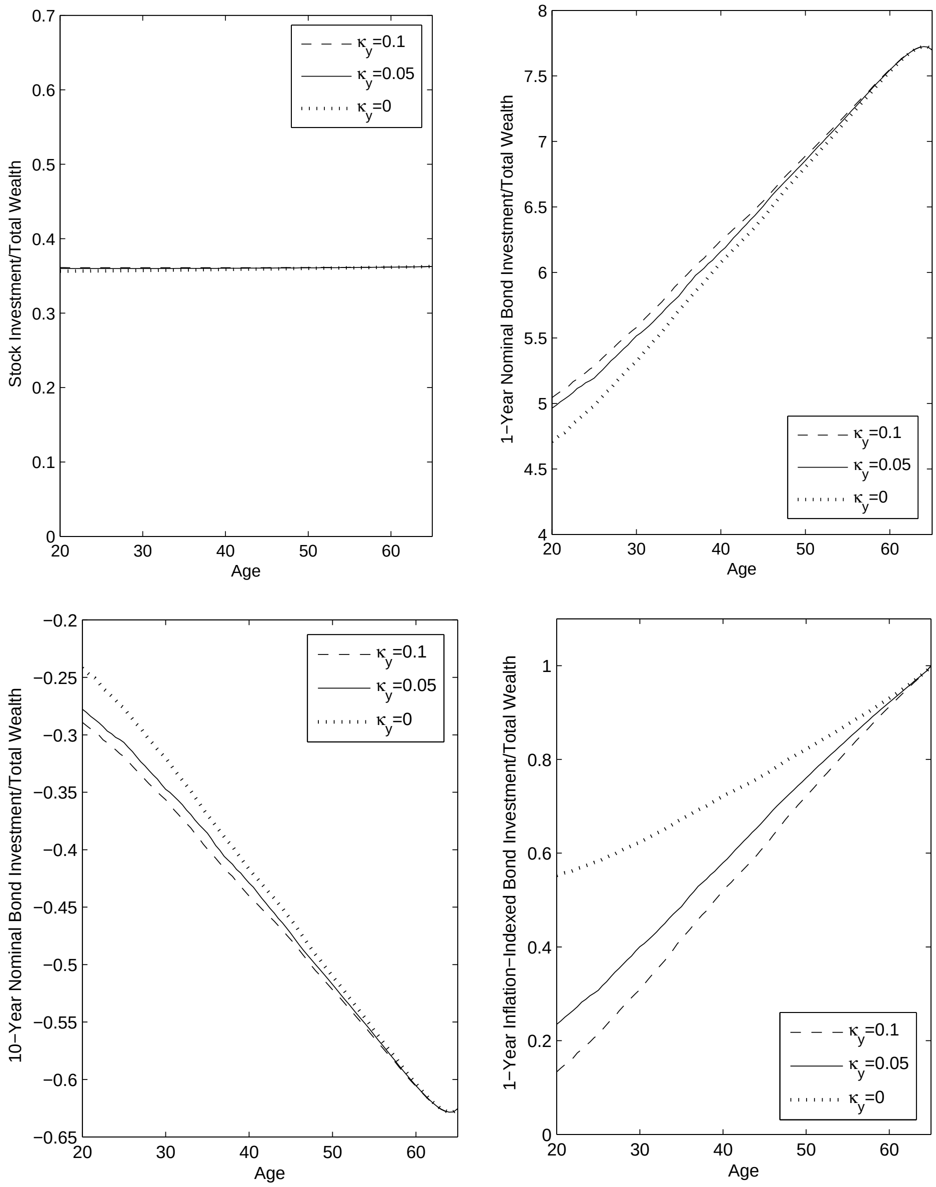


Figure 4.5: Optimal portfolio weights over the life-cycle in the case of partial wage rigidity ($\psi_\pi = 0.005$ and $\psi_u = 0.01$) . The figure shows how the optimal fractions of total wealth invested in the stock (top-left), the 1-year nominal bond (top-right), the 10-year nominal bond (bottom-left) and the 1-year inflation-indexed bond (bottom-right) evolve over the life-cycle in the case of partial wage rigidity where $\psi_\pi = 0.005$ and $\psi_u = 0.01$. The dotted curves depict the portfolio weights when $\kappa_y = 0$. The solid curves depict the portfolio weights when $\kappa_y = 0.05$. The dashed curves depict the portfolio weights when $\kappa_y = 0.1$. The graphs are drawn by taking averages over 1000 simulated paths using optimal strategies.

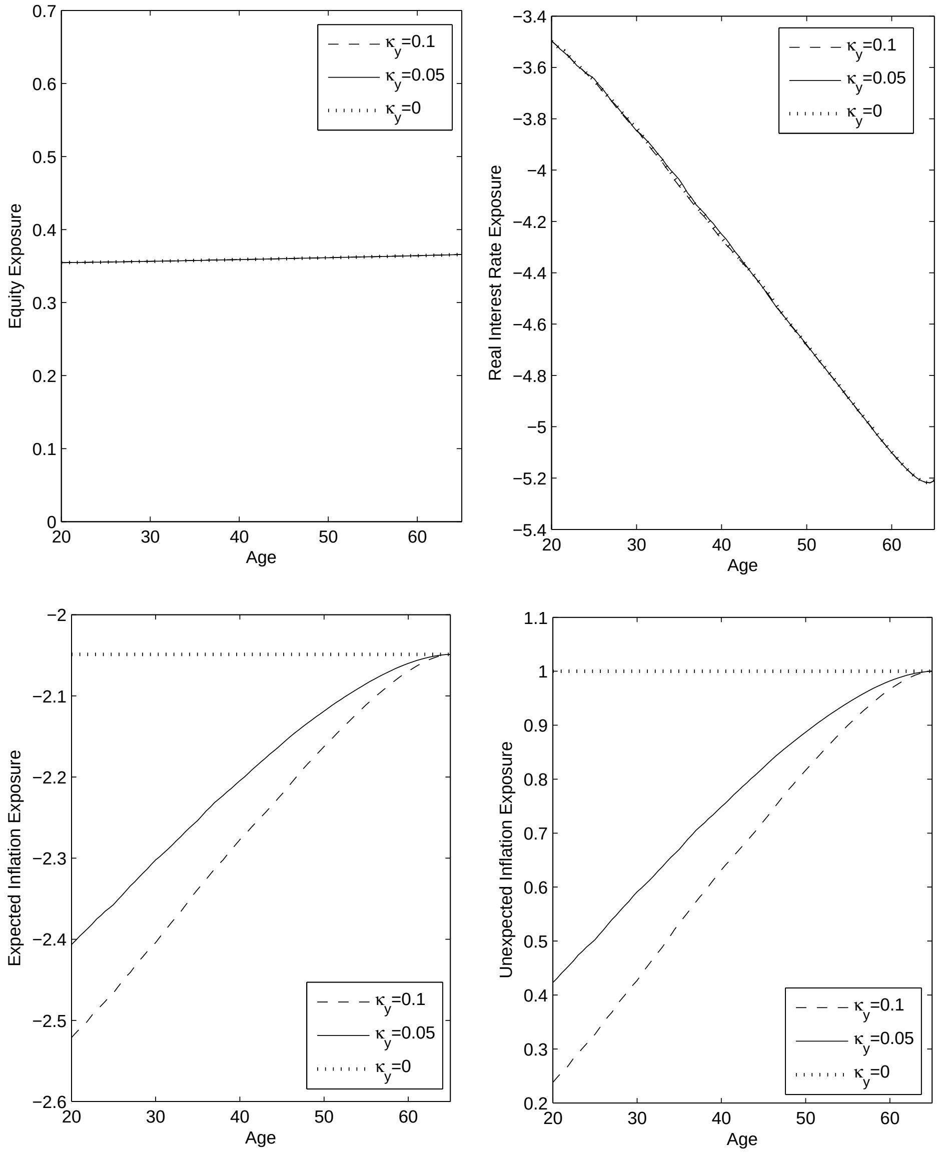


Figure 4.6: Optimal risk exposures over the life-cycle in the case of nominal wage rigidity ($\psi_\pi = 0$ and $\psi_u = 0$) . The figure shows how the equity exposure (top-left), the real interest rate exposure (top-right), the expected inflation risk exposure (bottom-left) and the unexpected inflation risk exposure (bottom-right) evolve over the life-cycle in the case of nominal wage rigidity where $\psi_\pi = 0$ and $\psi_u = 0$. The dotted curves depict the risk exposures when $\kappa_y = 0$. The solid curves depict the risk exposures when $\kappa_y = 0.05$. The dashed curves depict the risk exposures when $\kappa_y = 0.1$. The graphs are drawn by taking averages over 1000 simulated paths using optimal strategies.

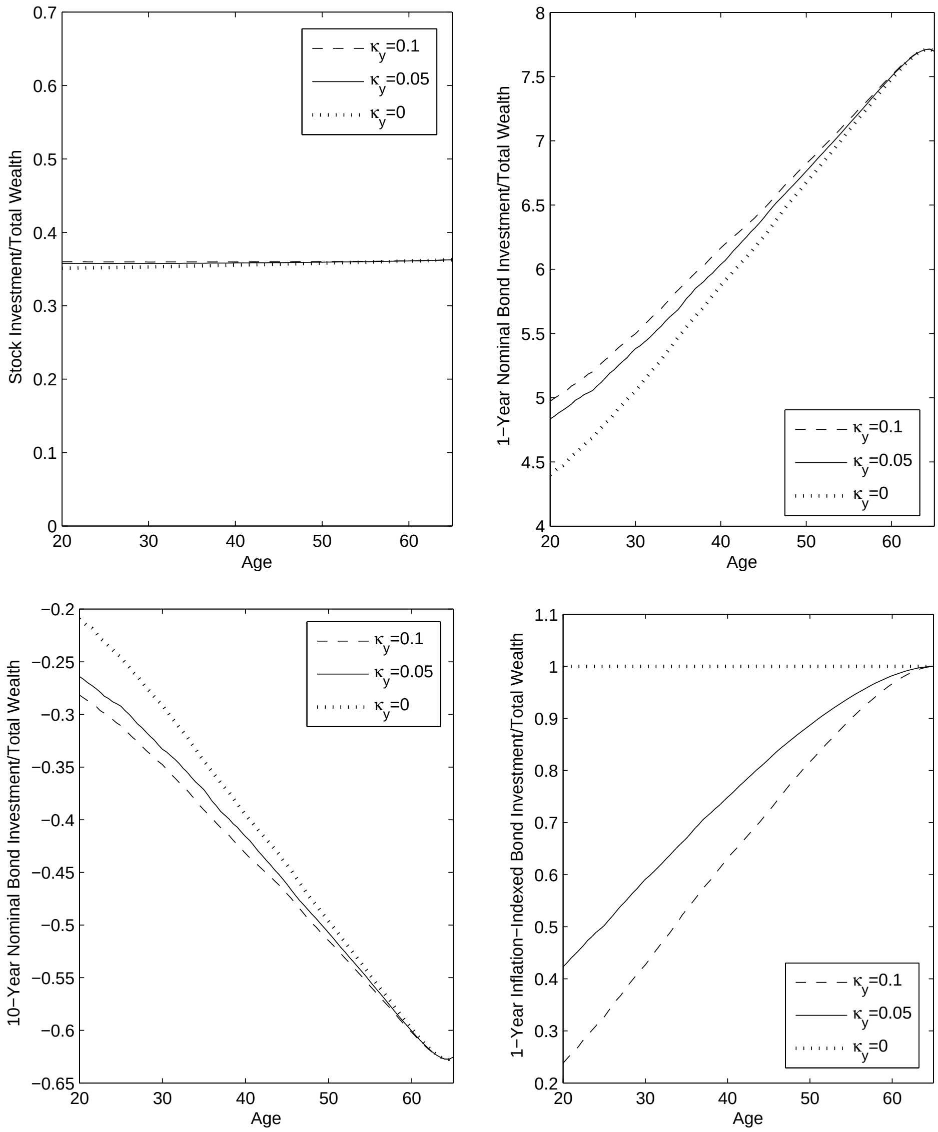


Figure 4.7: Optimal portfolio weights over the life-cycle in the case of nominal wage rigidity ($\psi_\pi = 0$ and $\psi_u = 0$). The figure shows how the optimal fractions of total wealth invested in the stock (top-left), the 1-year nominal bond (top-right), the 10-year nominal bond (bottom-left) and the 1-year inflation-indexed bond (bottom-right) evolve over the life-cycle in the case of nominal wage rigidity where $\psi_\pi = 0$ and $\psi_u = 0$. The dotted curves depict the portfolio weights when $\kappa_y = 0$. The solid curves depict the portfolio weights when $\kappa_y = 0.05$. The dashed curves depict the portfolio weights when $\kappa_y = 0.1$. The graphs are drawn by taking averages over 1000 simulated paths using optimal strategies.

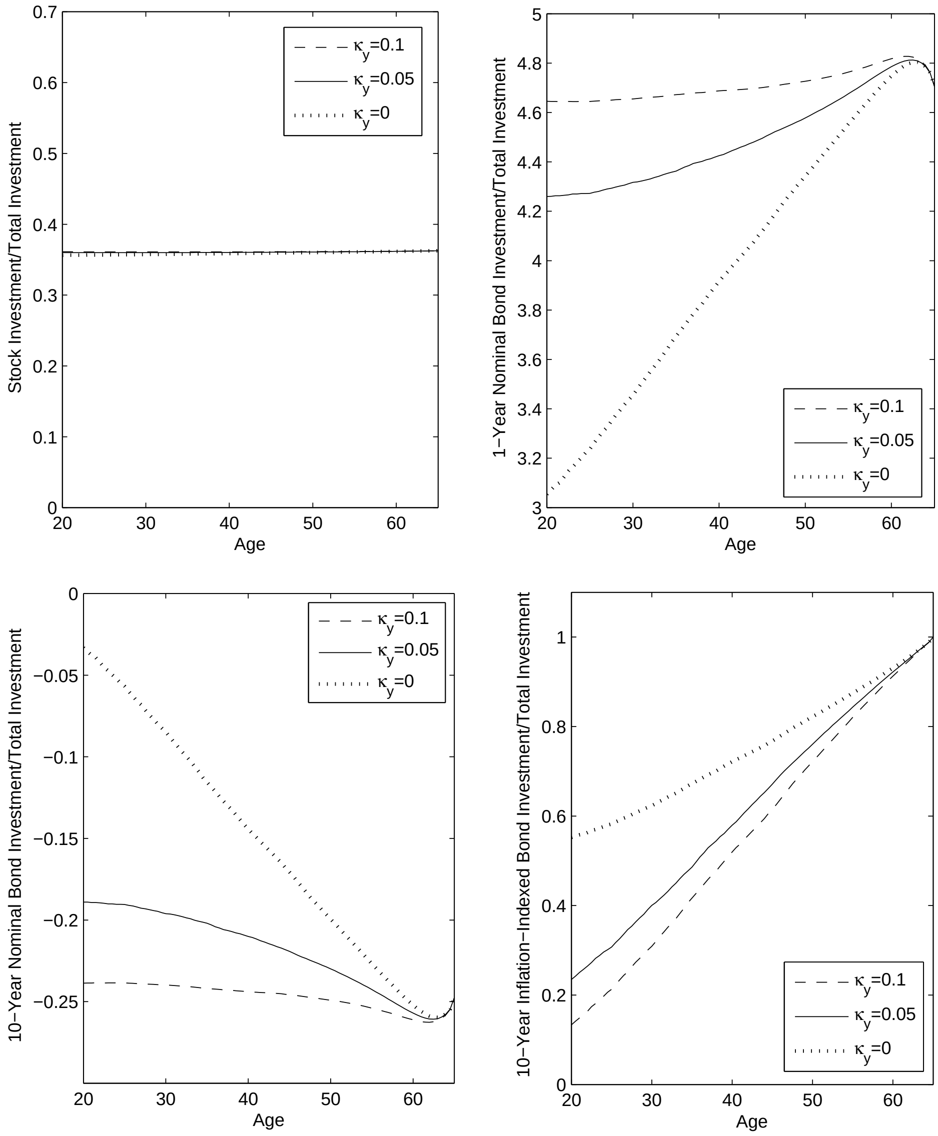


Figure 4.8: Optimal portfolio weights over the life-cycle in the case of partial wage rigidity ($\psi_\pi = 0$ and $\psi_u = 0$) when the maturity of the inflation-indexed bond is ten years. The figure shows how the optimal fractions of total wealth invested in the stock (top-left), the 1-year nominal bond (top-right), the 10-year nominal bond (bottom-left) and the 10-year inflation-indexed bond (bottom-right) evolve over the life-cycle in the case of nominal wage rigidity where $\psi_\pi = 0$ and $\psi_u = 0$. The dotted curves depict the portfolio weights when $\kappa_y = 0$. The solid curves depict the portfolio weights when $\kappa_y = 0.05$. The dashed curves depict the portfolio weights when $\kappa_y = 0.1$. The graphs are drawn by taking averages over 1000 simulated paths using optimal strategies.

optimal portfolio weights with a long-term inflation indexed bond is displayed in Figure 4.8. Comparison between Figure 4.5 and Figure 4.8 reveals that the allocations to the stock and the inflation-indexed bond stay unchanged. This is because the maturity of the inflation-indexed bond only affects its real interest rate exposure. In contrast, the absolute demands for the two nominal bonds becomes lower. This can be explained by the fact that the longer maturity leads to higher interest rate exposure provided by the holding of the inflation-indexed bond. Since the overall interest rate exposure is not affected by the choice of assets, that offered by the positions in the nominal bonds is reduced, which is responsible for the decline in the absolute demand for the nominal bonds.

4.5 Conclusion

In this chapter, we have considered a finite-horizon investor who receives exogenous labor income and maximizes her lifetime utility under inflation risk. Our model allows her aggregate labor income to be cointegrated with inflation. We have found empirical evidence in support of cointegration and shown that such a long-run relationship is crucial for the inflation-hedging property of human capital and the overall portfolio strategy.

We provide a closed-form solution for the valuation of human capital and the optimal portfolio and consumption strategy of an unconstrained investor in the absence of unspanned labor income risk. The cointegration relationship significantly raises the correlation between human capital and inflation and enhances the inflation hedging property of human capital because the negative impact of instantaneous inflation shocks on labor income is reduced by the rebound in its growth rate. To correct for the implicit investment via human capital, a new portfolio is set up, which depends on the riskiness of human capital and the relative importance of human capital in total wealth. We find that young investors' human capital effectively hedges inflation risk and substitutes for the long position in the inflation-indexed bond, but this crowding-out effect vanishes for older investors, because the ratio of human capital to total wealth declines and the cointegration relationship becomes irrelevant. This shows that inflation-indexed bonds are more important for the old than for the young.

Our analysis can be extended in several ways albeit with added computational complication. One extension would be to add residential real estate to the asset menu, because it is another major asset for many households, especially for older households and has been shown to provide an effective inflation hedge. The implicit inflation protection provided by housing would further undermine the importance of inflation-indexed bonds. Another extension is to explore alternative utility functions, such as habit formation. De Jong and Zhou (2013b) show that inflation risk may have substantial impact on the optimal strategy of habit-investors. However they do not consider labor income. Studying the effect of inflation risk on habit-investors in the working phase would be an interesting topic for future research.

4.6 Appendix

4.6.1 Calibration of the Term Structure Model

Following De Jong, Driessen, and Van Hemert (2008), we discretize our continuous-time term structure model as follows:

$$v_t = \beta_{v0} + \beta_{v1}(r_t - \bar{r}) + \beta_{v2}(\pi_t - \bar{\pi}) + u_{vt} \tag{4.49}$$

$$R_t = \beta_{R0} + (r_t - \bar{r}) + (\pi_t - \bar{\pi}) + u_{Rt} \tag{4.50}$$

$$\Delta \ln \Pi_{t+1} = \bar{\pi} + (\pi_t - \bar{\pi}) + \epsilon_{t+1} \tag{4.51}$$

$$r_t - \bar{r} = \beta_r(r_{t-1} - \bar{r}) + \eta_{rt} \tag{4.52}$$

$$\pi_t - \bar{\pi} = \beta_\pi(\pi_{t-1} - \bar{\pi}) + \eta_{\pi t} \tag{4.53}$$

where v, R and $\Delta \ln \Pi$ are the observable long-term bond yields, 3-month treasury bill rate and realized inflation. r and π are the unobserved real interest rate and expected inflation. The terms u_v and u_R are measurement errors, which are assumed to be i.i.d with mean zero and variance σ^2. The terms ϵ, η_r and η_π are discretized versions of $\sigma_\Pi \mathrm{d}z_\Pi$, $\sigma_r \mathrm{d}z_r$ and $\sigma_\pi \mathrm{d}z_\pi$. The parameters β_{v1}, β_{v2}, β_r and β_π are functions of the mean reversion parameters:

$$\beta_{v1} = (\kappa_r T)^{-1}\left(1 - e^{-\kappa_r T}\right), \qquad \beta_{v2} = (\kappa_\pi T)^{-1}\left(1 - e^{-\kappa_\pi T}\right) \tag{4.54}$$

where T is the bond maturity, and

$$\beta_r = \exp(-\kappa_r \Delta t), \qquad \beta_\pi = \exp(-\kappa_\pi \Delta t) \tag{4.55}$$

where Δt is the period of observations (0.25 for our quarterly observations).

We remove the intercepts β_{v0}, β_{R0} and $\bar{\pi}$ fitting them to the sample mean of the long-term bond yields, short rate and realized inflation. $\bar{r}$ and $\bar{\pi}$ need not to be estimated, because we use demeaned data on $r_t - \bar{r}$ and $\pi_t - \bar{\pi}$. Therefore, we end up with seven parameters to be estimated: (κ_r, κ_π, σ_r, σ_π, $\rho_{r\pi}$, σ_Π, σ). The estimation is performed by using the maximum likelihood method based on Kalman filter.

4.6.2 Proof of Proposition 1

Under the assumption of no labor income risks and no portfolio constraints, real human capital can be determined by discounting all future labor income streams:

$$h_t = \mathrm{E}_t\left[\int_t^T \frac{m_s}{m_t} l_s \mathrm{d}s\right] = l_t \int_t^T \mathrm{E}_t\left[\frac{m_s}{m_t}\frac{l_s}{l_t}\right] \mathrm{d}s \tag{4.56}$$

From the dynamics of m in (4.7) and l in (4.22), we get,

$$m_s = m_t \exp\left\{\int_t^s \left(-r - \frac{1}{2}\phi'\rho\phi\right) \mathrm{d}v + \int_t^s \phi' \mathrm{d}z\right\} \tag{4.57}$$

$$l_s = l_t \exp\left\{\int_t^s \left(-\kappa_y y_v + g_v\right) \mathrm{d}v + \int_t^s (\psi - \xi)' \mathrm{d}z\right\} \tag{4.58}$$

Hence,

$$\frac{m_s}{m_t}\frac{l_s}{l_t} = \exp\left\{\int_t^s \left(-r - \frac{1}{2}\phi'\rho\phi - \kappa_y y_v + g_v\right) \mathrm{d}v + \int_t^s (\phi + \psi - \xi)' \mathrm{d}z\right\}. \tag{4.59}$$

From the dynamics of y in (4.18), we can easily get

$$\mathrm{d}y_v = \mathrm{e}^{-\kappa_y(v-t)} y_t + \int_t^v \mathrm{e}^{-\kappa(v-s)} (\psi - \xi)' \mathrm{d}z. \tag{4.60}$$

Applying the Fubini rule for interchanging the order of integration yields

$$\int_t^s y_v \mathrm{d}v = B_y(t,s) y_t + \int_t^s B_y(v,s)(\psi - \xi)' \mathrm{d}z, \tag{4.61}$$

where B_y is given in equation (4.30). Plugging (4.61) into (4.59) and applying Itô's lemma, we obtain

$$\mathrm{E}_t\left[\frac{m_s}{m_t}\frac{l_s}{l_t}\right] = \mathrm{e}^{F(t,s) - \kappa_y B_y(t,s) y_t} p(t,s), \tag{4.62}$$

where $F(t,s)$ is given in equation (4.33). Integrating over s, we arrive at (4.29).

4.6.3 Proof of Proposition 2

We can write the Lagrangian for feasible consumption strategies as

$$\mathcal{L} = \mathrm{E}\left[\mathrm{e}^{-\delta t}\frac{(\frac{C_t}{\Pi_t})^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] - \omega\left\{\mathrm{E}\left[\int_0^T \frac{C_t}{\Pi_t}\mathrm{d}t\right] - \frac{W_0}{\Pi_0} - h_0\right\}. \tag{4.63}$$

The first-order condition implies

$$c_t = \mathrm{e}^{-\frac{\delta t}{\gamma}}\omega^{-\frac{1}{\gamma}}\left(\frac{m_t}{m_0}\right)^{-\frac{1}{\gamma}}. \tag{4.64}$$

Plugging (4.64) into the budget constraint (4.38), solving for x and eliminating x from (4.64) yields

$$c_t = \mathrm{e}^{-\frac{\delta t}{\gamma}}Q^{-1}(0, r_0)\left(\frac{m_t}{m_0}\right)^{-\frac{1}{\gamma}}(w_0 + h_0). \tag{4.65}$$

where $Q(0, r_0)$ and $\alpha(\tau)$ are given in equations (4.40) and (4.41).

Plugging c_t into budget, we obtain

$$w_t + h_t = \mathrm{e}^{-\frac{\delta t}{\gamma}}\left(\frac{m_s}{m_t}\right)^{-\frac{1}{\gamma}}Q^{-1}(0, r_0)Q(t, r_t)(w_0 + h_0), \tag{4.66}$$

$$= Q(t, r_t)c_t. \tag{4.67}$$

The stochastic term of (w_t+h_t) comes from $(m_s/m_t)^{-\frac{1}{\gamma}}$ and $Q(t, r_t)$. Thus, the optimally invested total wealth process is

$$\mathrm{d}(w_t + h_t) = \mathrm{d}BV + (w_t + h_t)\left(\frac{1-\gamma}{\gamma}\bar{B}_r(t, r_t)\sigma_r e_2 - \frac{\phi}{\gamma}\right)'\mathrm{d}z_t, \tag{4.68}$$

where BV is short for some bounded variation process and can be different in each occurrence of abbreviation.

On the other hand, from nominal wealth process in (4.26), we can derive real wealth process

$$\mathrm{d}w_t = \mathrm{d}BV + w_t(\sigma' x_t - \xi)'\mathrm{d}z_t, \tag{4.69}$$

Taking the derivative of (4.29), we obtain the dynamics of real human capital h, The dynamics of real human capital are

$$\mathrm{d}h_t = \mathrm{d}BV + h_t \left[b_{1t}(\psi - \xi) - b_{2t}\sigma_r e_2\right]' \mathrm{d}z_t, \tag{4.70}$$

where b_{1t} and b_{2t} are given in equations (4.35) and (4.36). Combining (4.69) and (4.70), we get

$$\mathrm{d}(w_t + h_t) = \mathrm{d}BV + \left[\sigma' x w_t - \xi w_t + h_t b_{1t}(\psi - \xi) - h_t b_{2t}\sigma_r e_2\right]' \mathrm{d}z_t, \tag{4.71}$$

Equating the stochastic terms of equations in (4.68) and (4.71) yields the the factor loadings of nominal financial wealth on the four risk factors, K^W,

$$\begin{aligned} K_t^W = \sigma_{vol}^{-1}\sigma' x_t = \sigma_{vol}^{-1}\left(\frac{1-\gamma}{\gamma}\bar{B}_r(t, r_t)\sigma_r e_2 - \frac{\phi}{\gamma} + \xi\right)\left(1 + \frac{h_t}{w_t}\right) \\ + \sigma_{vol}^{-1}\left[b_{1t}\psi + (1 - b_{1t})\,\xi - b_{2t}\sigma_r e_2\right]\frac{h_t}{w_t}. \end{aligned} \tag{4.72}$$

Scaling K^W by $w_t/(w_t + h_t)$, we obtain the portfolio risk exposures, K. Depending on the asset menu characterized by σ, we further determine the optimal portfolio weights, θ. The results are shown in (4.43) and (4.45).

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